# Equal Energy Directions

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An optimization problem:
Let $\underrightarrow{X_1}, \cdots, \underrightarrow{X_n} \in \mathbb{R}^d$ denote a mean 0 data set, with $n > d$.
Let $U = \left\{\underrightarrow{u_1}, \cdots, \underrightarrow{u_d}\right\}$ denote an orthonormal basis of $\mathbb{R}^d$.
For $j = 1, \cdots, d$ define the *Energy in the direction* $\underrightarrow{u_j}$ to be:

$$E_j = \sum_{i=1}^{n} \left(\underrightarrow{u_j}^t \underrightarrow{X_n}\right)^2 .$$

Also define the *average energy* to be:

$$\bar{E} = \frac{1}{d} \sum_{i=1}^{d} E_j .$$

The main goal is to find the orthonormal basis $U$ that gives approximately equal energy directions in the sense that:

$$\arg\min_U \sum_{j=1}^{d} \left(E_j - \bar{E}\right)^2 .$$

Conjecture: the minimum value of $\sum_{j=1}^{d} \left(E_j - \bar{E}\right)^2 = 0$ is always achievable.

This seems easy to show in the case of $d = 2$, since the total energy, $E_1 + E_2$ is a constant over all choices of $U$, and the difference $E_1 - E_2$ is maximized (and is $> 0$) by $U$ that makes $\underrightarrow{u_1}$ the PC1 direction and is mininized by $U$ (and is $< 0$) by $U$ that makes $\underrightarrow{u_2}$ the PC2 direction (assuming the data do not have constant energy in all directions).

If this conjecture is right, then this could perhaps be formulated as an equation instead of a minimization.