Sliding Ellipsoid

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Consider a standard ellipsoid $\mathcal{E}_p$ in $\mathbb{R}^p$, where $p \geq 2$, which is represented as

$$\mathcal{E}_p := \left\{ x \in \mathbb{R}^p : x^T Q^{-1} x \leq 1 \right\}, \tag{1}$$

where Q is a symmetric positive definite matrix. We find the intersection of this ellipsoid $\mathcal{E}_p$ and a given hyperplane in $\mathbb{R}^p$:

$$\mathcal{H}_p := \left\{ x \in \mathbb{R}^p : a^T x = \alpha \right\}, \tag{2}$$

where $a \neq 0$ and α are given. Without loss of generality, assume that a_p is nonzero, we can write $x_p = \frac{\alpha}{a_p} - \frac{a_1}{a_p} x_1 - \cdots - \frac{a_{p-1}}{a_p} x_{p-1}$. We define a linear transformation to map from $\mathbb{R}^{p-1}$ to $\mathbb{R}^p$ as

$$x = Pz + b = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ -a_1/a_p & -a_2/a_p & \cdots & -a_{p-1}/a_p \end{bmatrix} z + b$$

where $z := (x_1, x_2, \cdots, x_{p-1})^T$, and $b := (0, 0, \cdots, 0, \alpha/a_p)^T \in \mathbb{R}^p$. In this case, x belongs to $\mathcal{H}_p$. To find the intersection $\mathcal{E}_p \cap \mathcal{H}_p$, we have

$$x^T Q^{-1} x = (Pz + b)^T Q^{-1} (Pz + b) = z^T P^T Q^{-1} P z + 2 b^T Q^{-1} P z + b^T Q^{-1} b \leq 1.$$

Let $H := P^T Q^{-1} P$ which is symmetric and positive definite since $P \in \mathbb{R}^{p \times (p-1)}$ is full column rank. We can rewrite $x^T Q^{-1} x$ as

$$x^T Q^{-1} x = (z - s)^T H (z - s) + b^T Q^{-1} b - s^T H^{-1} s \leq 1,$$

where $s := H^{-1/2} P^T Q^{-1} b$. Hence, this turns out to be an ellipsoid of the form

$$(z - s)^T H (z - s) \leq r := 1 + s^T H^{-1} s - b^T Q^{-1} b,$$

as long as $r > 0$. This ellipsoid is in $\mathbb{R}^{p-1}$.