Rate Bounded Conservative Cascades.

Let $\{M_n, n \geq 0\}$ be a sequence of iid random variables with support in [0,1]. (These are the generators of the cascade.) We assume that M_n and $1 - M_n$ are identically distributed. Let

$$\text{bang}(x, a, b) = \begin{cases} \min(a,b) & \text{if } x \leq \min(a,b) \\ x & \text{if } \min(a,b) < x < \max(a,b) \\ \max(a,b) & \text{if } x \geq \max(a,b) \end{cases} .$$

Let $0 \leq b \leq 1$ and $b = .b_1 b_2 b_3 ...$ be the binary decimal representation of b. The value of the rate function at b after n steps of the rate-bounded cascade recursion be $r^n(b)$. Suppose the maximum rate is $R > 1$. Then we have:

$$r^0(b) = 1;$$

$$r^{n+1}(b) = 2r^n(b) M_{n+1}^{*(1-b_{n+1})}(1 - M_{n+1}^*)^{b_{n+1}}$$

where

$$M_{n+1}^* = \text{bang}(M_{n+1}, \frac{R}{2r^n(b)}, 1 - \frac{R}{2r^n(b)}).$$

Limiting distribution of $r^n(b)$

For a fixed $b \in [0,1]$ $\{r^n(b), n \geq 0\}$ is a Markov chain with state space $[0, R]$. The states 0 and R are absorbing. State R can be attained in a finite number of steps. However, it is not possible to reach state 0 in a finite number of steps. Thus in the limit $r^n(b)$ is either 0 or R. The symmetry of the distribution of M_n and the conservative nature of the cascade implies that the limiting distribution of $r^n(b)$ is independent of b and is given by

$$\lim_{n\to\infty} P(r^n(b) = R | r^0(b) = r) = \frac{r}{R}, \;\; 0 \leq b \leq 1, \;\; 0 \leq r \leq R.$$

$$\lim_{n\to\infty} P(r^n(b) = 0 | r^0(b) = r) = 1 - \frac{r}{R}, \;\; 0 \leq b \leq 1, \;\; 0 \leq r \leq R.$$

Limiting joint distribution of $r^n(a)$ and $r^n(b)$

The joint distribution of $r^n(a)$ and $r^n(b)$ is more complicated. Suppose $a \neq b$ and $.a_1 a_2 a_3 ...$, and $.b_1 b_2 b_3 ...$ are the binary decimal representation of a and b. Let

$$k = \min\{n \geq 1 : a_n \neq b_n\}.$$

Since $a \neq b$, we have $k < \infty$. Then $r^n(a)$ and $r^n(b)$ are dependent for $n = 1, ..., k$. For $n \geq 1$, $r^{k+n}(a)$ and $r^{n+k}(b)$ are conditionally independent, given $r^k(a)$ and $r^k(b)$. Indeed, we have

$$\lim_{n\to\infty} P(r^{k+n}(a) = R, r^{k+n}(a) = R | r^k(a), r^k(b)) = \frac{r^k(a)}{R} \cdot \frac{r^k(b)}{R},$$

$$\lim_{n\to\infty} P(r^{k+n}(a)=0, r^{k+n}(a)=R | r^k(a), r^k(b)) = (1-\frac{r^k(a)}{R})\frac{r^k(b)}{R},$$

$$\lim_{n\to\infty} P(r^{k+n}(a)=R, r^{k+n}(a)=0 | r^k(a), r^k(b)) = \frac{r^k(a)}{R}(1-\frac{r^k(b)}{R}),$$

$$\lim_{n\to\infty} P(r^{k+n}(a)=0, r^{k+n}(a)=0 | r^k(a), r^k(b)) = (1-\frac{r^k(a)}{R})(1-\frac{r^k(b)}{R}).$$

Thus the study of the limiting joint distribution reduces to that of the transient distribution. We illustrate with an example.

Case: $0 \le a < .5 \le b < 1$

In this case $a_1 = 0$ and $b_1 = 1$ and hence $k = 1$. From the recursion we have:

$$r^1(a) = 2M_1^*, \quad r^1(b) = 2(1-M_1^*).$$

Hence,

$$\lim_{n\to\infty} P(r^{n+1}(a)=R, r^{n+1}(a)=R | r^1(a)=2M_1^*, r^k(b)=2(1-M_1^*)) = \frac{2M_1^*}{R}\cdot\frac{2(1-M_1^*)}{R}.$$

Unconditioning, we get

$$P(r(a)=R, r(b)=R) = \frac{4}{R^2}E(M_1^*(1-M_1^*)).$$

When $M_1 \sim U(0,1)$ this reduces to

$$P(r(a)=R, r(b)=R) = \begin{cases} \frac{-4+11R-4R^2}{3R^2} & \text{if } 1 \le R \le 2 \\ \frac{1}{6} & \text{if } R > 2 \end{cases}.$$

Hence,

$$Cov(r(a), r(b)) = \begin{cases} -\frac{(R-1)(4R-7)}{3} & \text{if } 1 \le R \le 2 \\ -\frac{1}{3} & \text{if } R > 2 \end{cases}.$$

It will be interesting to compute $Cov(r^n(a), r^n(b))$ for $0 \le a \le b \le 1$. This will tell us what kind of autocorrelation structure to expect from a rate-bounded conservative cascade.

Similar recursions can be derived for the inverse rate-bounded conservative cascade.