IS NETWORK TRAFFIC APPROXIMATED BY STABLE LÉVY MOTION OR FRACTIONAL BROWNIAN MOTION?

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Abstract. Cumulative broadband network traffic is often thought to be well modelled by fractional Brownian motion. However, some traffic measurements do not show an agreement with the Gaussian marginal distribution assumption. We show that if connection rates are modest relative to heavy tailed connection length distribution tails, then stable Lévy motion is a sensible approximation to cumulative traffic over a time period. If connection rates are large relative to heavy tailed connection length distribution tails, then FBM is the appropriate approximation. The results are framed as limit theorems for a sequence of cumulative input processes whose connection rates are varying in such a way as to remove or induce long range dependence.

1. Introduction

Recent analysis of broadband measurements shows that the data sets exhibit three characteristic properties: heavy tails, self-similarity and long range dependence (LRD). Traditional traffic models using independent inter-arrival times with distribution tails which are exponentially bounded imply short range dependence in the traffic and hence are not appropriate for describing high-speed network traffic. Empirical evidence on the existence of self-similarity and LRD in traffic measurements can be found in [24, 9, 11]. A common explanation for observed LRD and self-similarity of network traffic is heavy tailed transmission times. Sometimes, this is due to file lengths being heavy tailed [8, 10, 12, 13, 11, 2] and sometimes due to heavy tailed burst lengths, where a burst is a period where packet arrivals are not separated by more than some threshold value [28].

Analysts are largely in agreement about the self-similar nature of aggregate traffic, at least at time scales above a certain threshold. Empirical [45, 2] and theoretical [40, 17, 18, 19] evidence supports the heavy tailed explanation of the self-similarity. However, measurement studies diverge in their conclusions about the marginal distributions of cumulative traffic. There exists empirical evidence supporting a heavy tailed assumption backed by theoretical work [15, 23, 32] and also spirited interest and evidence for Gaussian marginal distributions [25].

The point of this paper is to show that with heavy tailed input times, cumulative traffic at large time scales can look either heavy tailed or Gaussian depending on whether the rate at which transmissions are initiated (crudely referred to as the *connection rate*) is moderate or quite large.

There are two related models which frame this discussion:

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Model (i): the superposition of M ON/OFF sources (see for example [45, 40, 24, 19, 17, 18]), and

Model (ii): the infinite source Poisson model, sometimes called the M/G/∞ input model (see [19, 31, 1, 26, 21, 20, 37]).

In model (i), traffic is generated by a large number of independent ON/OFF sources such as workstations in a big computer lab. An ON/OFF source transmits data at a constant rate to a server if it is ON and remains silent if it is OFF. Every individual ON/OFF source generates an ON/OFF process consisting of independent alternating ON- and OFF-periods. The lengths of the ON-periods are identically distributed and so are the lengths of OFF-periods. Support for this model in the form of statistical analysis of Ethernet Local Area Network traffic of individual sources was provided in [45]; the conclusions of this study are that the lengths of the ON- and OFF-periods are heavy tailed and in fact Pareto-like with tail parameter between 1 and 2. In particular, the lengths of the ON- and OFF-periods have finite means but infinite variances. Further evidence is in [24] and [10]. The latter authors found evidence of Pareto-like tails in file lengths, transfer times and idle times in World Wide Web traffic; see also [11].

Model (ii), the infinite source Poisson model, assumes transmission initiations or *connections* by sources at times of a rate λ Poisson process. The transmission durations are iid random variables independent of the times of connection initiation. The transmission lengths have finite mean, infinite variance and heavy tails. During a transmission, a source transmits at unit rate.

For both models, the process we study is $A(t)$, the cumulative input in $[0,t]$ by all sources. Because both models assume unit rate transmissions, we may write

$$A(t) = \int_0^t N(s)\, ds, \quad t > 0 ,$$

where $N(s)$ is the number of active sources at time s. For large T, we think of $(A(Tt), t \geq 0)$ as the process on large time scales. Our results for both models show that if the connection rate is allowed to depend on T in such a way that it has a growth rate in T which is moderate (in a manner to be made precise), then $A(T\cdot)$ looks like an α-stable Lèvy motion, while if the connection rate grows faster than a critical value, $A(T\cdot)$ looks like a fractional Brownian motion.

Section 2 defines the models formally and Section 3 defines *slow* and *fast* growth for the connection rate. Subsequent sections show that for our models, *slow growth* means that cumulative input can be approximated by a stable Lévy motion while *fast growth* means cumulative input should be approximated by fractional Brownian motion.

2. Model formulation

We now define our two related models and give basic discussion.

2.1. **The ON/OFF model.** Consider first a single ON/OFF source such as a workstation as described in [18]. During an ON-period, the source generates traffic at a constant rate 1, e.g. 1 byte per time unit. During an OFF-period, the source remains silent; we assign the value 0 to it. Let $X_{\rm on}, X_1, X_2, \ldots$ be iid non-negative random variables representing the lengths of ON-periods and $Y_{\rm off}, Y_1, Y_2, \ldots$ be iid non-negative random variables representing the lengths of OFF-periods. We also write

$$Z_i = X_i + Y_i\,, \quad i \geq 0\,.$$

The X- and Y-sequences are supposed to be independent. For any distribution function F we write $\overline{F} = 1 - F$ for the right tail. By $F_{\rm on}/F_{\rm off}$ we denote the common distribution of ON/OFF-periods.

In what follows, we assume that

$$\overline{F}_{\rm on}(x) = x^{-\alpha_{\rm on}} L_{\rm on}(x) \quad \text{and} \quad \overline{F}_{\rm off}(x) = x^{-\alpha_{\rm off}} L_{\rm off}(x), \quad x > 0\,, \tag{2.1}$$

where $\alpha_{\rm on}, \alpha_{\rm off} \in (1,2)$ and $L_{\rm on}, L_{\rm off}$ are slowly varying at infinity. Hence, both distributions $F_{\rm on}$ and $F_{\rm off}$ have finite means $\mu_{\rm on}$ and $\mu_{\rm off}$ but their variances are infinite. Notice that the tail parameters $\alpha_{\rm on}$ and $\alpha_{\rm off}$ may be different, hence the extremes of the ON- and OFF-periods can differ significantly. For the purposes of this paper, we always assume that

$$\alpha := \alpha_{\rm on} < \alpha_{\rm off}\,. \tag{2.2}$$

Assuming (2.2) makes the results for Model (i) and Model (ii) almost identical. The case of general $\alpha_{\rm on}$ and $\alpha_{\rm off}$ can be treated in a similar way; see [27].

Consider the renewal sequence generated by the alternating ON- and OFF-periods (cf. [18]). Renewals happen at the beginnings of the ON-periods, the inter-arrival distribution is $F_{\rm on} * F_{\rm off}$ and the mean inter-arrival time

$$\mu = EZ_1 = \mu_{\rm on} + \mu_{\rm off}\,.$$

In order to make the renewal sequence stationary (see [35, page 224]), a delay random variable T_0 is introduced which is independent of the X_i's and the Y_i's. A stationary version of the renewal sequence (T_n) is then given by

$$T_0\,, \quad T_n = T_0 + \sum_{i=1}^{n} Z_i\,, \quad n \geq 1\,. \tag{2.3}$$

One way to construct the delay variable T_0 (see [18]) is as follows. Let B, $X_{\rm on}^{(0)}$ and $Y_{\rm off}^{(0)}$ be independent random variables, independent of $\{Y_{\rm off}, (X_n), (Y_n)\}$, such that B is Bernoulli with

$$P(B=1) = \mu_{\rm on}/\mu = 1 - P(B=0)\,,$$

and

$$P(X_{\rm on}^{(0)} \leq x) = \frac{1}{\mu_{\rm on}} \int_0^x \overline{F}_{\rm on}(s)\,ds =: F_{\rm on}^{(0)}(x)\,,$$

$$P(Y_{\rm off}^{(0)} \leq x) = \frac{1}{\mu_{\rm off}} \int_0^x \overline{F}_{\rm off}(s)\,ds =: F_{\rm off}^{(0)}(x)\,.$$

Define

$$T_0 = B(X_{\rm on}^{(0)} + Y_{\rm off}) + (1-B) Y_{\rm off}^{(0)}.$$

The renewal sequence (2.3) is then stationary.

The *ON/OFF process of one source* is now defined as the indicator process

$$W_t = B\, 1_{[0, X_{\rm on}^{(0)})}(t) + \sum_{n=0}^{\infty} 1_{[T_n, T_n + X_{n+1})}(t)\,, \quad t \geq 0\,. \tag{2.4}$$

The ON/OFF process W is a binary process with $W_t = 1$ if t is in an ON-period and $W_t = 0$ if t is in an OFF-period. The stationarity of the renewal sequence (2.3) implies strict stationarity of the process W with mean

$$EW_t = P(W_t = 1) = \mu_{\rm on}/\mu\,.$$

The precise rate of decay for $\gamma_W(h)$, the covariance function of the stationary process W, under the assumptions (2.1) and $\alpha_{\rm on} < \alpha_{\rm off}$ is given in [18]. As $h \to \infty$,

$$\gamma_W(h) \sim \frac{\mu_{\rm off}^2}{(\alpha - 1)\mu^3}\, h^{-(\alpha-1)} L_{\rm on}(h) = (\text{const})\, h \overline{F}_{\rm on}(h)\,. \tag{2.5}$$

The process W exhibits LRD in the sense that

$$\sum_k |\gamma_W(k)| = \infty\,, \tag{2.6}$$

See [3] for different definitions and discussion of LRD.

Now consider a superposition of M iid ON/OFF sources $(W_t^{(m)}, m = 1, \ldots, M; t \geq 0)$ feeding a server. The number of active sources at time t is

$$N(t) = N_M(t) = \sum_{m=1}^{M} W_t^{(m)}, \quad t \geq 0\,.$$

Note that $N(t)$ is the input rate to the server at time t and can be referred to as the *workload process.* Since the sources are iid, (2.5) implies that N exhibits LRD in the spirit of (2.6), since the stationary version of N satisfies

$$\gamma_N(h) = \sum_{i=1}^{M} \gamma_{W^{(i)}}(h) = (\text{const}) M h \overline{F}_{\text{on}}(h).$$

The *cumulative input* of work to the server or *total accumulated work* by time t is

$$A(t) = A_M(t) = \int_0^t N(s)\, ds, \quad t \geq 0\,. \tag{2.7}$$

The behavior of the cumulative input process $A(t)$ for the superposition of a large number of iid ON/OFF sources has been studied in [45, 40] where it was found that the cumulative input process (properly normalized) of an increasing number of iid ON/OFF sources converges to fractional Brownian motion in the sense of convergence of the finite dimensional distributions. Their result is formulated as a double limit: first, the number M of sources goes to infinity and then the time-scaling parameter T converges to infinity. This order of taking limits is crucial for obtaining fractional Brownian motion as limit. When limits are taken in reversed order, the limits of the finite dimensional distributions are those of infinite variance stable Lévy motion. The increment process of fractional Brownian motion, fractional Gaussian noise, exhibits LRD reflecting the LRD in the original workload process. This is in contrast to stable Lévy motion, which while self-similar, has increments which are independent.

In [45, 40] a double limit is involved and the limit regime is sequential. This sequential procedure is unsatisfactory both theoretically and in practice. (Similar remarks are made in [22].) The limiting behavior of the cumulative input process depends on the relative sizes of the number of sources M, the time-scaling parameter T and the tail probabilities of the transmission lengths. We study simultaneous limit regimes, in which both M and T go to infinity at the same time. We assume that $M = M_T$ goes to infinity as $T \to \infty$. The ON/OFF models change as $T \to \infty$, and we will refer to the *Tth model.* The number of sources $M = M_T$ plays the role of the *connection rate.*

2.2. **The infinite source Poisson model.** Here is the precise formulation of this model.

Let $(\Gamma_k, -\infty < k < \infty)$ be the points of a rate λ homogeneous Poisson process on $\mathbb{R}$, labeled so that $\Gamma_0 < 0 < \Gamma_1$ and hence $\{-\Gamma_0, \Gamma_1, (\Gamma_{k+1} - \Gamma_k, k \neq 0)\}$ are iid exponentially distributed random variables with parameter λ. The random measure which counts the points is denoted by $\sum_{k=-\infty}^{\infty} \epsilon_{\Gamma_k}$ and is a Poisson random measure (PRM) with mean measure $\lambda\mathbb{L}$, where $\mathbb{L}$ stands for Lebesgue measure. We imagine that a communication system has an infinite number of nodes or *sources*, and at time Γ_k a connection is made and some node begins a transmission at constant rate to the server. As a normalization, this constant rate is taken to be unity. The lengths of transmissions are random variables X_k. We assume $X_{\rm on}, X_1, X_2, \dots$ are iid and independent of (Γ_k) and

$$P(X_{\rm on} > x) = \overline{F}_{\rm on}(x) = x^{-\alpha} L(x), \quad x > 0,\ 1 < \alpha < 2\,, \tag{2.8}$$

where L is a slowly varying function. Since $\alpha \in (1,2)$, the variance of $X_{\rm on}$ is infinite and its mean $\mu_{\rm on}$ is finite. We will need the quantile function

$$b(t) = \left(1/\overline{F}_{\rm on}\right)^{\leftarrow}(t), \quad t > 0\,, \tag{2.9}$$

which is regularly varying with index $1/\alpha$. We note that

$$\nu = \sum_{k=-\infty}^{\infty} \epsilon_{(\Gamma_k, X_k)}, \tag{2.10}$$

the counting function on $\mathbb{R} \times [0,\infty]$ corresponding to the points $\{(\Gamma_k, X_k)\}$, is a two dimensional Poisson process on $\mathbb{R} \times [0,\infty]$ with mean measure $\lambda\mathbb{L} \times F_{\rm on}$; cf. [34].

The first quantity of interest is $N(t)$, the number of active sources at time t, which has representation

$$N(t) = N_T(t) = \sum_{k=-\infty}^{\infty} 1_{[\Gamma_k \le t < \Gamma_k + X_k]} = \nu(\{(s,y) \in \mathbb{R} \times (0,\infty] : s \le t < s + y\}). \tag{2.11}$$

The notation N_T refers to the fact that we will consider a family of Poisson processes indexed by the scaling parameter $T > 0$ such that the intensity $\lambda = \lambda(T)$ goes to infinity as $T \to \infty$. Under this condition, the considered models change when time goes by, and therefore we will refer to the *Tth model.* The intensity $\lambda = \lambda(T)$ will be referred to as the *connection rate.*

The second expression in (2.11) makes it clear that for each t, $N(t)$ is a Poisson random variable with parameter

$$\begin{aligned} &\lambda\mathbb{L} \times F_{\rm on}(\{(s,y) \in \mathbb{R} \times (0,\infty] : s \le t < s + y\}) \\ &\quad = \int_{s=-\infty}^{t} \int_{y=t-s}^{\infty} \lambda\mathbb{L}(ds) \times F_{\rm on}(dy) = \lambda \int_{-\infty}^{t} \overline{F}_{\rm on}(t-s)ds = \lambda\, \mu_{\rm on}\,. \end{aligned} \tag{2.12}$$

During a transmission, the transmitting node is sending data to the server at unit rate. The *total accumulated input in* $[0,t]$ for the Tth model is

$$A(t) = A_T(t) = \int_0^t N(s)\, ds. \tag{2.13}$$

Analogous to (2.5), we find that heavy tailed transmission times X_k induce LRD in N. By means of a point process argument dating to Cox [7] we can show that

$$\mathrm{Cov}(N(t), N(t+h)) = \lambda \int_h^{\infty} \overline{F}_{\rm on}(v)\, dv \sim (\mathrm{const})\, h\, \overline{F}_{\rm on}(h) = (\mathrm{const})\, h^{-(\alpha-1)}\, L(h), \tag{2.14}$$

as $h \to \infty$. Once again we see that high variability in transmission times causes LRD in the rate at which work is offered to the system.

3. The critical input rate

Recall the measures of dependence given by (2.5) and (2.14). We will find that *cumulative input is well approximated by stable Lévy motion*, a process with independent increments, *when the connection rate is slow*, or equivalently when dependence in the Tth model disappears as $T \to \infty$, while *fractional Brownian motion is the appropriate approximation when the connection rate is fast* or dependence in the Tth model remains strong as $T \to \infty$. In what follows, we make precise what a "fast" or "slow" connection rate means in both the infinite source Poisson model and the superposition of ON/OFF processes. The definitions for the two models are virtually identical apart from obvious changes in notation.

3.1. The infinite source Poisson model.

Recall that $\lambda = \lambda(T)$ is the parameter governing the connection rate in the Tth model and suppose $\lambda = \lambda(T)$ is a non-decreasing function of T. We phrase our condition first in terms of the quantile function b defined in (2.9). The asymptotic behavior of $A_T(\cdot)$ depends on whether

$$\text{Slow Growth Condition 1:} \quad \lim_{T\to\infty} \frac{b(\lambda T)}{T} = 0,$$

or

$$\text{Fast Growth Condition 2:} \quad \lim_{T\to\infty} \frac{b(\lambda T)}{T} = \infty$$

holds. Notice that $b(\cdot)$ is regularly varying with index $1/\alpha$.

The next lemma provides an alternate way to express the conditions.

Lemma 1. *Assume $F_{\rm on}$ satisfies* (2.8). *Consider the stationary version of the input rate $N_T(\cdot)$.*

1. *The slow growth condition* 1 *is equivalent to any of the two conditions*

$$\lim_{T\to\infty} \lambda T\, \overline{F}_{\rm on}(T) = 0 \quad or \quad \lim_{T\to\infty} \ {\rm Cov}(N_T(0), N_T(T)) = 0\,.$$

2. *The fast growth condition* 2 *is equivalent to any of the two conditions*

$$\lim_{T\to\infty} \lambda T\, \overline{F}_{\rm on}(T) = \infty \quad or \quad \lim_{T\to\infty} {\rm Cov}(N_T(0), N_T(T)) = \infty\,.$$

Remark. The interpretation of the two conditions in terms of the LRD is nicely observed in [41].

Proof. In the case of Condition 1, there exists a function $0 < \epsilon(T) \to 0$ such that $T\epsilon(T) \to \infty$ and $b(\lambda T) = T\epsilon(T)$. Thus

$$\lambda T \sim 1/\overline{F}_{\rm on}(T\epsilon(T))\,. \tag{3.1}$$

Therefore, Condition 1 implies

$$\lambda T\, \overline{F}_{\rm on}(T) \sim \overline{F}_{\rm on}(T)/\overline{F}_{\rm on}(T\epsilon(T)) \ \to 0\,. \tag{3.2}$$

Conversely, if $\delta(T) := \lambda T \overline{F}_{\rm on}(T) \to 0$, then using $b^{\leftarrow}(T) \sim 1/\overline{F}_{\rm on}(T)$, we get

$$\frac{b(\lambda T)}{T} \sim \frac{b(\delta(T) b^{\leftarrow}(T))}{b(b^{\leftarrow}(T))} \to 0\,,$$

and so Condition 1 and (3.2) are equivalent. Similarly, Condition 2 is the same as

$$\lambda T\, \overline{F}_{\rm on}(T) \to \infty\,. \tag{3.3}$$

To get the equivalence in terms of the covariances, use (2.14) and

$$\int_T^\infty \overline{F}_{\text{on}}(s)\, ds \sim (\text{const})\, T\overline{F}_{\text{on}}(T)\,,$$

which follows from Karamata's theorem for regularly varying functions; see [5]. □

We will see that if the rate of increase of λ satisfies Condition 1, then $A(T)$ is asymptotically a stable random variable while in the alternate case, it is asymptotically normal.

For the work in later sections, proofs are expedited by the following fact.

Lemma 2. *If Condition* 1 *holds, then*

$$\lim_{T\to\infty} \frac{\lambda T^2 \overline{F}_{\text{on}}(T)}{b(\lambda T)} = 0, \tag{3.4}$$

and if Condition 2 *holds, this limit is infinite.*

Proof. Assume that Condition 1 holds. As with (3.1), set $\epsilon(T) = b(\lambda T)/T \to 0$ so that $\epsilon(T)T \to \infty$. Denoting the ratio in (3.4) by $r(T)$, we see that

$$r(T) \sim \frac{\overline{F}_{\text{on}}(T)}{\epsilon(T)\overline{F}_{\text{on}}(T\epsilon(T))}\,,$$

and using the Karamata representation of a regularly varying function (see [5]), we obtain

$$r(T) \sim [\epsilon(T)]^{-1} \exp\left\{ - \int_{T\epsilon(T)}^{T} u^{-1}\, \alpha(u)\, du \right\} \tag{3.5}$$

for some function $\alpha(u) \to \alpha$, as $u \to \infty$. Since $1 < \alpha < 2$, we may pick δ so small that $\alpha - \delta > 1$ and since $T\epsilon(T) \to \infty$, we have for F sufficiently large, that the right-hand side in (3.5) is bounded from above by

$$[\epsilon(T)]^{-1} \exp\left\{-(\alpha-\delta)\ \log\left(1/\epsilon(T)\right)\right\} = [\epsilon(T)]^{\alpha-\delta-1}\,,$$

and the right-hand side converges to zero as $T \to \infty$. The proof of an infinite limit under Condition 2 is similar. □

3.2. **The ON/OFF model.** Recall the ON/OFF model from Section 2.1. In analogy with the infinite source Poisson model, it is possible to introduce a slow and a fast growth condition in terms of the number $M = M(T)$ of ON/OFF processes. Assume that $M = M(T)$ is some integer-valued function such that

$$M(T) \text{ is non-decreasing in } T \text{ and } \lim_{T\to\infty} M(T) = \infty.$$

For ease of presentation we usually suppress the dependence of M on T.

The role of the Poisson intensity $\lambda = \lambda(T) \to \infty$ is now played by the number $M = M(T) \to \infty$ of the ON/OFF sources. As in the former case we introduce growth conditions on $M = M(T)$. For the slow growth condition we again use the quantile function b of F_{on} introduced in (2.9). The asymptotic behavior of the cumulative workload $A = A_M$ of M iid sources will depend on whether

$$\text{Slow Growth Condition 1:} \quad \lim_{T\to\infty} \frac{b(MT)}{T} = 0,$$

or

$$\text{Fast Growth Condition 2:} \quad \lim_{T\to\infty} \frac{b(MT)}{T} = \infty$$

holds. These conditions are directly comparable to the critical connection rate for the infinite source Poisson model. For later use, observe that Lemmas 1 and 2 hold provided M is substituted for λ.

The next two sections discuss why α-stable Lévy motion is the appropriate limit under slow growth. We begin by studying this result in the slightly simpler context of the infinite source Poisson model.

4. α–STABLE APPROXIMATIONS FOR THE INFINITE SOURCE POISSON MODEL UNDER SLOW GROWTH

In this section we assume Condition 1 holds and show why A is asymptotically an α-stable Lévy motion.

Recall, for example from [38], that a continuous in probability process $(X_{\alpha,\sigma,\beta}(t)\,, t \geq 0)$ with stationary, independent increments and càdlàg sample paths is called α-stable Lévy motion if $X_{\alpha,\sigma,\beta}(t) \sim S_\alpha(\sigma t^{1/\alpha}, \beta, 0)$. Here $S_\alpha(\sigma, \beta, \mu)$ denotes the α-stable distribution which is characterized by the index of stability $\alpha \in (0, 2]$, the scale parameter $\sigma \geq 0$, the skewness parameter $\beta \in [-1, 1]$ and the shift parameter $\mu \in \mathbb{R}$. If $X \sim S_\alpha(\sigma, \beta, \mu)$, then its characteristic function is given by

$$E\mathrm{e}^{\,i\theta X} = \begin{cases} \exp\Big\{ -\sigma^\alpha\,|\theta|^\alpha\,(1 - i\beta\,\mathrm{sign}(\theta)\,\tan(\pi\alpha/2)) + i\mu\theta \Big\} & \text{if } \alpha \neq 1, \\ \exp\Big\{ -\sigma\,|\theta| \left(1 + i\beta\,\dfrac{2}{\pi}\,\mathrm{sign}(\theta)\,\ln|\theta|\right) + i\mu\theta \Big\} & \text{if } \alpha = 1. \end{cases}$$

The case $\alpha = 2$ corresponds to the Gaussian distribution. Notice that $X_{2,\sigma,\beta}$ is Brownian motion, whereas $\alpha < 2$ implies that $X_{\alpha,\sigma,\beta}$ has infinite variance marginal distributions. In contrast to Brownian motion which has continuous sample paths with probability 1, infinite variance stable Lévy motion has discontinuous sample paths with probability 1.

4.1. The main result.

The following theorem is our main result under the slow growth condition.

Theorem 1. *If Condition 1 holds, then the process $(A(Tt), t \geq 0)$ describing the total accumulated input in $[0, Tt]$, $t \geq 0$, satisfies the limit relation*

$$\frac{A(T\cdot) - T\lambda\mu_{\mathrm{on}}(\cdot)}{b(\lambda T)} \stackrel{fidi}{\to} X_{\alpha,1,1}(\cdot)\,.$$

Here $\stackrel{fidi}{\to}$ denotes convergence of the finite dimensional distributions.

Remark. The convergence can be extended to M_1 convergence by following the proof in [33]. The papers of Whitt [43, 44, 42] are enlightening. The convergence cannot be extended to J_1 convergence in the Skorokhod space $\mathbb{D}$. This follows, for example, from [23] who show that a sequence of processes with a.s. continuous sample paths cannot converge in distribution in $(\mathbb{D}, J_1)$ to a process with a.s. discontinuous sample paths. Our favorite reference to weak convergence in metric spaces is [4].

In the rest of this section we give the proof of Theorem 1.

4.2. The basic decomposition.

We start by giving a useful decomposition of the random variable $A(T)$. In the sequel we frequently suppress the dependence on T in the notation of the considered quantities.

Let

$$
\begin{aligned}
R_1 &:= \{(s,y) : 0 < s \le T,\ 0 < y,\ s+y \le T\}, \\
R_2 &:= \{(s,y) : 0 < s \le T,\ T < s+y\}, \\
R_3 &:= \{(s,y) : s \le 0,\ 0 < s+y \le T\}, \\
R_4 &:= \{(s,y) : s \le 0,\ T < s+y\},
\end{aligned} \tag{4.1}
$$

and rewrite (2.13) as

$$
\begin{aligned}
A(T) &= \sum_k X_k 1_{[(\Gamma_k, X_k)\in R_1]} + \sum_k (T-\Gamma_k) 1_{[(\Gamma_k, X_k)\in R_2]} + \sum_k (X_k+\Gamma_k) 1_{[(\Gamma_k, X_k)\in R_3]} \\
&\quad + \sum_k T 1_{[(\Gamma_k, X_k)\in R_4]} \\
&=: A_1 + A_2 + A_3 + A_4 .
\end{aligned} \tag{4.2}
$$

Recall the definition of the PRM ν from (2.10) with mean measure $\lambda\mathbb{L} \times F_{\text{on}}$. Note that A_i is a function of the points of ν in region R_i, and since the R_i's are disjoint, $A_i, i = 1, \ldots, 4$, are mutually independent. Calculations as in (2.12) and use of Karamata's theorem give that as $T \to \infty$

$$
\begin{aligned}
\lambda m_1 := E\nu(R_1) &= \lambda \int_0^T F_{\text{on}}(T-s)\, ds \sim \lambda T, \\
\lambda m_2 := E\nu(R_2) &= \lambda \int_0^T \overline{F}_{\text{on}}(T-s)\, ds \sim \lambda\, \mu_{\text{on}}, \\
\lambda m_3 := E\nu(R_3) &= \lambda \int_{s=-\infty}^0 \int_{y=-s}^{-s+T} F_{\text{on}}(dy)\, ds \sim \lambda \mu_{\text{on}} \\
\lambda m_4 := E\nu(R_4) &= \lambda \int_{s=-\infty}^0 \int_{-s+T}^{\infty} F_{\text{on}}(dy)\, ds = \lambda \int_T^\infty \overline{F}_{\text{on}}(u) du \\
&\sim \lambda T \overline{F}_{\text{on}}(T)/(\alpha-1) \to 0.
\end{aligned} \tag{4.3}
$$

So the mean measure $E\nu(\cdot)$ restricted to R_i is finite for $i = 1, \ldots, 4$, which implies that the points of $\nu\big|_{R_i}$ can be represented as a Poisson number of iid random vectors:

$$
\nu\big|_{R_i} \stackrel{d}{=} \sum_{k=1}^{P_i} \epsilon_{(t_{k,i}, j_{k,i})}, \quad i = 1, \ldots, 4,
$$

where P_i is a Poisson random variable with mean λm_i, which is independent of the iid pairs $(t_{k,i}, j_{k,i})$, $k \ge 1$, with common distribution

$$
\frac{\lambda \mathbb{L}(ds) F_{\text{on}}(dy)}{\lambda m_i}\Big|_{R_i} = \frac{\mathbb{L}(ds) F_{\text{on}}(dy)}{m_i}\Big|_{R_i}, \tag{4.4}
$$

for $i = 1, \ldots, 4$. Notice that the distributions of $((t_{k,i}, j_{k,i}))$ are independent of λ, which only enters into the specification of the mean of P_i, $i = 1, \ldots, 4$. This means that for fixed T, we can represent the A_i's as sums of a Poisson number of iid random variables,

$$
\begin{aligned}
A_1 &\stackrel{d}{=} \textstyle\sum_{k=1}^{P_1} j_{k,1}, & A_2 &\stackrel{d}{=} \textstyle\sum_{k=1}^{P_2} (T - t_{k,2}), \\
A_3 &\stackrel{d}{=} \textstyle\sum_{k=1}^{P_3} (j_{k,3} + t_{k,3}), & A_4 &\stackrel{d}{=} \textstyle\sum_{k=1}^{P_4} T.
\end{aligned} \tag{4.5}
$$

4.3. **Moments of the summands.** In what follows, we will need information about the moments of the summands in (4.5). All the variables are bounded by T so all moments exist, but we need to know the asymptotic form of the moments as $T \to \infty$. For notational ease, we will as before frequently suppress the dependence on T in the notation used, and let (t_i, j_i) be random variables with the same distribution as $(t_{k,i}, j_{k,i})$, for $i = 1, \dots, 4$.

¿From (4.3) and (4.4), observe that for $l \geq 1$,

$$
\begin{aligned}
Ej_1^l &= \int_0^T \int_{y=0}^{T-s} y^l \frac{\mathbb{L}(ds)F_{\rm on}(dy)}{m_1} \sim \frac{1}{T}\int_{s=0}^T \int_{y=0}^{T-s} y^l \, F_{\rm on}(dy)\, ds \\
&= \frac{1}{T}\int_{s=0}^T \left(\int_{y=0}^{s} y^l F_{\rm on}(dy) \right) ds. \qquad (4.6)
\end{aligned}
$$

For $l = 1$, since $\int_0^s yF_{\rm on}(dy) \to \mu_{\rm on}$, we have

$$
Ej_1 \to \mu_{\rm on}\,. \qquad (4.7)
$$

For $l > \alpha$, we have, using first a change of variables and then Karamata's theorem, that

$$
\frac{Ej_1^l}{T^l \overline{F}_{\rm on}(T)} \sim \int_0^1 \int_0^s y^l \, \frac{F_{\rm on}(Tdy)}{\overline{F}_{\rm on}(T)} \, ds \sim \int_0^1 \int_0^s y^l \, \alpha y^{-1-\alpha} \, dy \, ds = \frac{\alpha}{(l-\alpha)(l-\alpha+1)}. \qquad (4.8)
$$

For later reference we note that (4.7) and (4.8) imply

$$
\frac{\mathrm{Var}(j_1)}{T^2 \overline{F}_{\rm on}(T)} \sim \int_0^1 \int_0^s y^2 \, \alpha y^{-1-\alpha} \, dy \, ds \; = \frac{\alpha}{(2-\alpha)(3-\alpha)} =: \; \sigma_1^2 \qquad (4.9)
$$

and

$$
\limsup_{T\to\infty} \frac{E|j_1 - Ej_1|^3}{T^3 \overline{F}_{\rm on}(T)} \leq \limsup_{T\to\infty} \frac{4\Big(Ej_1^3 + \big(Ej_1\big)^3\Big)}{T^3 \overline{F}_{\rm on}(T)} = \mathrm{const}\,. \qquad (4.10)
$$

Similar calculations for $T - t_2$, give that for $l \geq 1$,

$$
\begin{aligned}
E(T-t_2)^l &= \int_{s=0}^T \int_{y=T-s}^{\infty} (T-s)^l \frac{\mathbb{L}(ds)F_{\rm on}(dy)}{m_2} \sim \frac{1}{\mu_{\rm on}} \int_{s=0}^T \int_{y=T-s}^{\infty} (T-s)^l \, F_{\rm on}(dy)\, ds \\
&= \frac{1}{\mu_{\rm on}} \int_0^T u^l \overline{F}_{\rm on}(u) du,
\end{aligned}
$$

and therefore, for $l \geq 1$, as $T \to \infty$, from Karamata's theorem

$$
\begin{aligned}
\frac{E(T-t_2)^l}{T^{l+1}\overline{F}_{\rm on}(T)} &\sim \frac{1}{\mu_{\rm on}} \int_0^1 x^l \frac{\overline{F}_{\rm on}(Tx)}{\overline{F}_{\rm on}(T)} dx \\
&\sim \frac{1}{\mu_{\rm on}} \int_0^1 x^{l-\alpha} dx = \frac{1}{\mu_{\rm on}(l-\alpha+1)}. \qquad (4.11)
\end{aligned}
$$

This implies that

$$
\frac{\mathrm{Var}(T-t_2)}{T^3 \overline{F}_{\rm on}(T)} \sim \frac{1}{\mu_{\rm on}(3-\alpha)} =: \; \sigma_2^2 \qquad (4.12)
$$

and

$$
\limsup_{T\to\infty} \frac{E|T - t_2 - E(T-t_2)|^3}{T^4 \overline{F}_{\rm on}(T)} \leq \mathrm{const}\,. \qquad (4.13)
$$

Finally,

$$E(j_3+t_3)^l = \int_{s=-\infty}^{0}\int_{y=-s}^{T-s}(y+s)^l\frac{\mathbb{L}(ds)F_{\rm on}(dy)}{m_3} \sim \frac{1}{\mu_{\rm on}}\int_{-\infty}^{0}\int_{y=-s}^{T-s}(y+s)^l\,F_{\rm on}(dy)\,ds\,,$$

and thus

$$\frac{E(j_3+t_3)^l}{T^{l+1}\overline{F}_{\rm on}(T)} \sim \frac{1}{\mu_{\rm on}}\int_{s=-\infty}^{0}\int_{y=-s}^{1-s}(y+s)^l\,\frac{F_{\rm on}(Tdy)}{\overline{F}_{\rm on}(T)}\,ds$$
$$\sim \frac{1}{\mu_{\rm on}}\int_{s=-\infty}^{0}\int_{y=-s}^{1-s}(y+s)^l\,\alpha y^{-1-\alpha}\,dy\,ds. \tag{4.14}$$

It follows that

$$\frac{\mathrm{Var}(j_3+t_3)}{T^3\overline{F}_{\rm on}(T)} \sim \frac{1}{\mu_{\rm on}}\int_{s=-\infty}^{0}\int_{y=-s}^{1-s}(y+s)^2\,\alpha y^{-1-\alpha}\,dy\,ds =: \sigma_3^2$$

and

$$\limsup_{T\to\infty}\frac{E|j_3-t_3-E(j_3-t_3)|^3}{T^4\overline{F}_{\rm on})T)} \le \text{const}\,.$$

To compute σ_3^2, observe that

$$\begin{aligned}
\sigma_3^2 :=& \frac{1}{\mu_{\rm on}}\int_{s=0}^{\infty}\int_{y=s}^{1+s}(y-s)^2\alpha y^{-\alpha-1}dyds\\
=&\frac{1}{\mu_{\rm on}}\int_{y=0}^{1}\alpha y^{-\alpha-1}\Big[\int_{s=0}^{y}(y-s)^2ds\Big]dy\\
&+\frac{1}{\mu_{\rm on}}\int_{y=1}^{\infty}\alpha y^{-\alpha-1}\Big[\int_{s=y-1}^{y}(y-s)^2ds\Big]dy\\
=&\frac{1}{\mu_{\rm on}}\int_{y=0}^{1}\alpha y^{-\alpha-1}\Big[\frac{y^3}{3}\Big]dy+\frac{1}{\mu_{\rm on}}\int_{y=1}^{\infty}\alpha y^{-\alpha-1}\Big[\frac{1}{3}\Big]dy\\
=&\frac{1}{\mu_{\rm on}}\Big[\frac{\alpha}{3(3-\alpha)}+\frac{1}{3}\Big]=\frac{1}{\mu_{\rm on}(3-\alpha)}.
\end{aligned} \tag{4.15}$$

4.4. **α-stable limits: one dimensional convergence.** We are now in position to show that if Condition 1 holds, then $A(T)$ is asymptotically an α-stable random variable. The plan is to show $A_1(T)=A_1$ is asymptotically stable and $A_i(T)=A_i$, $i=2,3,4$ are asymptotically negligible. As before we frequently suppress the dependence on T in the notation.

It is relatively easy to see that

$$A_i/b(\lambda T)\stackrel{P}{\to}0\,,\quad i=2,3,4\,. \tag{4.16}$$

We restrict ourselves to the case $i=2$; a similar argument works for $i=3,4$. By (4.11), Lemma 2 and Condition 1,

$$EA_2 = EP_2\,E(T-t_2) = [\lambda m_2]\,E(T-t_2) \sim (\text{const})\,\lambda T^2\overline{F}_{\rm on}(T) = o(b(\lambda T))\,.$$

Thus it remains to consider A_1. Recall the representation of A_1 given in (4.5). We start with the following decomposition:

$$A_1-\lambda\mu_{\rm on}T = \sum_{k=1}^{P_1}(j_{k,1}-Ej_1)+Ej_1\,[P_1-EP_1]+[EA_1-\lambda\mu_{\rm on}T] = A_{11}+A_{12}+A_{13}\,.$$

By (4.7), $Ej_1 \sim \mu_{\rm on}$. Since P_1 is Poisson with mean $\lambda m_1 \to \infty$, it satisfies the central limit theorem, i.e.

$$[\lambda m_1]^{-1/2}\,[P_1 - \lambda m_1] \xrightarrow{d} N(0,1)\,. \tag{4.17}$$

We conclude that

$$A_{12} = O_P([\lambda T]^{1/2}) = o_P(b(\lambda T))\,. \tag{4.18}$$

By (4.5) and (4.17), A_{11} is a sum of approximately $\lambda m_1 \sim \lambda T$ iid summands. Under Condition 1, $b(\lambda T)/T \to 0$, so that for any $x > 0$ fixed, we eventually have $T - b(\lambda T)x > 0$. Therefore, from (4.4)

$$\begin{aligned}
\lambda T P(j_1 > b(\lambda T)\,x) &= \iint_{\substack{0\le s\le T\\ 0\le s+y\le T\\ y>b(\lambda T)x}} \frac{ds F_{\rm on}(dy)}{m_1} = \int_{s=0}^{T-b(\lambda T)x} ds \int_{y=b(\lambda T)x}^{t-s} \frac{F_{\rm on}(dy)}{m_1}\\
&= \frac{1}{m_1}\overline{F}_{\rm on}(b(\lambda T)x)(T-b(\lambda T)x) - \frac{1}{m_1}\int_0^{t-b(\lambda T)x} \overline{F}_{\rm on}(T-s)ds\\
&\sim \left(1-\frac{b(\lambda T)x}{T}\right)\ \lambda T\,\overline{F}_{\rm on}(b(\lambda T)\,x) - \frac{b(\lambda T)}{T}\int_x^{T/b(\lambda T)} \lambda T\,\overline{F}_{\rm on}(b(\lambda T)\,s)\,ds\\
&\sim x^{-\alpha}\,.
\end{aligned}$$

Following a standard argument using point processes ([34, Exercise 4.4.2.8, page 222], [31]), we get for $t \ge 0$,

$$X^{(T)}(t) := [b(\lambda T)]^{-1}\sum_{k=1}^{[\lambda T t]}(j_{k,1} - Ej_1) \xrightarrow{d} X_{\alpha,1,1}(t) \quad \text{in } \mathbb{D}[0,\infty), \tag{4.19}$$

where the limit is a totally skewed α-stable Lévy random motion. In fact, by independence, we may couple (4.17) and (4.19) to get joint convergence

$$\left(X^{(T)}(\cdot), \frac{P_1}{\lambda T}\right) \xrightarrow{d} (X_{\alpha,1,1}(\cdot), 1) \quad \text{in } \mathbb{D}[0,\infty)\times\mathbb{R}.$$

Using composition and the continuous mapping theorem, one obtains

$$[b(\lambda T)]^{-1}A_{11} = X^{(T)}(P_1/(\lambda T)) = [b(\lambda T)]^{-1}\sum_{i=1}^{P_1}(j_{k,1} - Ej_1) \xrightarrow{d} X_{\alpha,1,1}(1)\,. \tag{4.20}$$

It remains to consider A_{13}. By (4.6) and Karamata's theorem,

$$\begin{aligned}
A_{13} &= Ej_1\,EP_1 - \lambda T\,\mu_{\rm on} = \lambda \int_0^T \left[\int_0^s y\,F_{\rm on}(dy) - \mu_{\rm on}\right] ds\\
&= -\lambda \int_0^T \int_s^\infty y\,F_{\rm on}(dy)\,ds\\
&\sim -(\text{const})\,\lambda T^2 \overline{F}_{\rm on}(T) = o(b(\lambda T))\,. \qquad (4.21)
\end{aligned}$$

The last limit relation follows from Lemma 2. Combining the limit relations (4.16), (4.18), (4.20) and (4.21), we conclude that $A(T)$ has the desired α-stable limit.

4.5. **α–stable limits: finite dimensional convergence.** We restrict ourselves to showing the convergence of the 2-dimensional distributions; the general case is completely analogous. Suppose $t_1 < t_2$. The same arguments as for the one dimensional convergence show that it suffices to consider the joint convergence of $[b(\lambda T)]^{-1}(A_1(Tt_i) - \lambda Tt_i\ \mu_{\text{on}})$, $i = 1, 2$. We can write

$$\begin{aligned} A_1(Tt_2) &= A_1(Tt_1) + \sum_{Tt_1<\Gamma_k\le Tt_2} X_k 1_{[\Gamma_k+X_k\le Tt_2]} + \sum_{\Gamma_k\le Tt_1} X_k 1_{[Tt_1<\Gamma_k+X_k\le Tt_2]} \\ &=: A_1(Tt_1) + A_{21}(T(t_2-t_1)) + A_{22}\,. \end{aligned}$$

Observe that $A_1(Tt_1)$ and $A_{21}(T(t_2-t_1))$ are independent and that $A_{21}(T(t_2-t_1)) \stackrel{d}{=} A_1(T(t_2-t_1))$. Hence the proof of the 2-dimensional distributions follows from the 1-dimensional convergence if one can show that $[b(\lambda T)]^{-1}A_{22} \stackrel{P}{\to} 0$. However,

$$\begin{aligned} EA_{22} &= E\left(\iint_{0\le s\le Tt_1, Tt_1<s+u\le Tt_2} u\ \nu(ds,du)\right) \\ &= \lambda \iint_{0\le s\le Tt_1, Tt_1<s+u\le Tt_2} u\ ds\ F_{\text{on}}(du) \\ &= \lambda \int_{s=0}^{Tt_1}\left(\int_{u=Tt_1-s}^{Tt_2-s} u\ F_{\text{on}}(du)\right)\ ds \\ &= \lambda T^2 \overline{F}_{\text{on}}(T)\ \int_0^{t_1}\left(\int_{u=t_1-s}^{t_2-s} u\frac{F_{\text{on}}(Tdu)}{\overline{F}_{\text{on}}(T)}\right)\ ds \\ &\sim \lambda T^2 \overline{F}_{\text{on}}(T)\ \left[\int_0^{t_1} \frac{\alpha}{\alpha-1}\Big((t_1-s)^{-(\alpha-1)} - (t_2-s)^{-(\alpha-1)}\Big)\ ds\right] \\ &= o(b(\lambda T))\,, \end{aligned}$$

by Lemma 2. This concludes the proof of Theorem 1. □

5. α–STABLE APPROXIMATIONS FOR THE SUPERPOSITION OF ON/OFF PROCESSES UNDER SLOW GROWTH

Recall the ON/OFF model from Section 2.1 and the slow and fast growth conditions on $M = M_T$ from Section 3.2.

5.1. **The main result for $\alpha := \alpha_{\text{on}} < \alpha_{\text{off}}$.** In this section we show that the cumulative input process $(A(Tt), t \ge 0)$ as introduced in (2.7) has a limiting α-stable Lévy motion provided the slow growth Condition 1 holds, i.e. $b(MT) = o(T)$.

The following theorem is our main result on the weak convergence of the process A. To formulate it, we will need the following notation:

$$c := \mu_{\text{off}}/\mu^{1+1/\alpha} \quad \text{and} \quad \sigma := C_\alpha^{-1/\alpha}\,,$$

where

$$C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}\,. \tag{5.1}$$

Theorem 2. *If Slow Growth Condition* 1 *holds then the process* $(A(Tt), t \geq 0)$ *describing the cumulative input in* $[0, Tt], t \geq 0$, *satisfies the limit relation*

$$\frac{A(T\cdot) - TM\mu^{-1}\mu_{\text{on}}(\cdot)}{b(MT)} \stackrel{fidi}{\rightarrow} c\, X_{\alpha,\sigma,1}(\cdot)$$

where $\stackrel{fidi}{\rightarrow}$ *denotes convergence of the finite dimensional distributions and* $X_{\alpha,\sigma,1}$ *is an* α*-stable Lévy motion as described at the beginning of Section 4.*

Remarks. 1) Theorems 1 and 2 have some striking similarities. Both results yield α-stable Lévy motions in the limit under slow growth conditions which are also directly comparable in terms of the quantile function of F_{on}. Moreover, the normalizations in both results were defined in a similar way.
2) As for the infinite source Poisson model, the convergence cannot be extended to functional convergence in $(\mathbb{D}[0,\infty), J_1)$ (although it has been claimed in the literature) since the J_1-limiting process of a sequence of processes with a.s. continuous sample paths should have a.s. continuous sample paths. An alternative proof of the impossibility of J_1-convergence, using an extreme value argument, was given in [39].
3) The case of general $\alpha_{\text{on}}, \alpha_{\text{off}} \in (1,2)$ is treated in [27]. The results are qualitatively the same, yielding a limiting $\min(\alpha_{\text{on}}, \alpha_{\text{off}})$-stable Lévy motion. Moreover, the skewness parameter of the limit process may vary between -1 and $+1$, depending on the right tails of F_{on} and F_{off}.

In the rest of this section we give the proof of our main result. It will be convenient to split the proof into different parts.

5.2. **The basic decomposition.** As for the proof of Theorem 1, we give a decomposition of the cumulative input process. This decomposition is very much in the spirit of Section 4.2. In what follows, we will adapt the notation of Section 2.1 for the mth source. Whenever we consider only one source we will suppress the dependence on m in the notation.

Recall the construction of a stationary version of the renewal sequence (T_n) given in (2.3). We consider the renewal sequence $(T_n^{(m)})$ corresponding to the mth source and define the corresponding renewal counting process

$$\xi_T^{(m)} := \sum_{n=0}^{\infty} 1_{[0,T]}(T_n^{(m)}) \quad \text{with mean } \mu_T = E\xi_T^{(m)} = T/\mu.$$

For convenience, we also write

$$Z_i^{(m)} = X_i^{(m)} + Y_i^{(m)}, \quad i \geq 1\,.$$

We have the following basic decomposition of $A(T)$ for M iid sources (cf. (2.4)):

$$\begin{aligned} A(T) &= \sum_{m=1}^{M} B^{(m)} \min(T, (X_{\text{on}}^{(0)})^{(m)}) + \sum_{m=1}^{M} \sum_{k=1}^{\xi_T^{(m)}} X_k^{(m)} - \sum_{m=1}^{M} \max(0, T^{(m)}_{\xi_T^{(m)}-1} + X^{(m)}_{\xi_T^{(m)}} - T) 1_{[\xi_T^{(m)} \geq 1]} \\ &=: A_1 + A_2 + A_3\,. \end{aligned}$$

Remark. The above decomposition of $A(T)$ is similar to the infinite source Poisson model; see (4.2). The crucial difference is that, for every m, the counting process $\xi_T^{(m)}$ is heavily dependent on the sequence $(X_k^{(m)})$ which appears in the random sum representation of A_2. This fact makes the proof below more complicated. The basic idea of the proof consists of replacing the counting processes $\xi_T^{(m)}$ in A_2 simultaneously by their identical means μ_T. After the replacement, the resulting process is a sum of iid random variables and so classical limit theory for sums of iid random variables comes in. The replacement described above is provided by a large deviation result given in the Appendix.

5.3. A_1 **and** A_3 **are asymptotically negligible.** We show under the slow growth condition on M that the terms A_1 and A_3 vanish in the limit. The case A_1 is relatively easy.

Lemma 3. *As* $T \to \infty$,

$$[b(MT)]^{-1}(A_1 - EA_1) \xrightarrow{P} 0\,.$$

Proof. We have

$$[b(MT)]^{-1} EA_1 \le M\ E \min(T, X_{\rm on}^{(0)})$$

Using Karamata's theorem, we obtain

$$M\ [b(MT)]^{-1} \int_0^T P(X_{\rm on}^{(0)} > x)\, dx \quad \le \quad ({\rm const})\ M\ [b(MT)]^{-1}\, T^{2-\alpha}\, L_{\rm on}(T) \to 0\,. \tag{5.2}$$

In the last step we used Lemma 2. □

In the rest of the section we show that A_3 is asymptotically negligible. By virtue of the slow growth condition $b(MT) = o(T)$, we can find a function $\epsilon_T \to 0$ such that

$$b(MT) = o(\epsilon_T\, T) \quad \text{and} \quad 1/\log(T) = o(\epsilon_T) \quad \text{as } T \to \infty. \tag{5.3}$$

For example, we could let

$$\epsilon_T = \left(\frac{b(MT)}{T}\right)^{1/2} \bigvee \frac{1}{\sqrt{\log T}}.$$

Lemma 4. *Assume that* ϵ_T *satisfies* (5.3). *Then the relation*

$$M\ P(|\xi_T - \mu_T| > \epsilon_T\ \mu_T) = o(1) \quad as\ T \to \infty$$

holds.

Proof. First we treat the case $\xi_T > (1+\epsilon_T)\mu_T$. Since $Z_i = X_i + Y_i$ has a regularly varying right tail there exist iid mean-zero random variables E_i concentrated on $[-EZ_1, \infty)$ and a positive number x_0 such that for some $\beta > 0$

$$P(Z_1 - EZ_1 > x) \ge P(E_1 > x) \quad \text{for } x \ge -EZ_1 \text{ and } P(E_1 > x) = {\rm e}^{-\beta x}\,,\ \ x \ge x_0\,.$$

Then a stochastic domination argument shows that with $m_T = [(1+\epsilon_T)\mu_T]$,

$$\begin{aligned} P(\xi_T > (1+\epsilon_T)\mu_T) &= P(T_0 + Z_1 + \cdots + Z_{m_T} \le T) \\ &\le P(Z_1 + \cdots + Z_{m_T} - m_T\mu \le T - m_T\mu) \\ &\le P(E_1 + \cdots + E_{m_T} \le T - m_T\mu) \\ &= P((m_T {\rm Var}(E_1))^{-1/2}(E_1 + \cdots + E_{m_T}) \le -a_T) =: p(T)\,, \end{aligned}$$

where

$$a_T := (m_T {\rm Var}(E_1))^{-1/2}\,(m_T\mu - T)\,.$$

Since $\mu_T = T/\mu$, we have for some $|\theta_T| \le 1$, that

$$a_T \sim ({\rm const})\frac{\epsilon_T T + \theta_T}{\sqrt{T}} \sim ({\rm const})\epsilon_T T^{1/2},$$

and hence for all large T

$$a_T \geq T^{1/6}\,.$$

The classical Cramér result on large deviations for sums of iid random variables with moment generating function existing in a neighborhood of the origin (see [29, Theorem 3, Chapter VIII]) gives for large T,

$$\begin{aligned} p(T) &\leq P((m_T \mathrm{Var}(E_1))^{-1/2}(E_1+\cdots+E_{m_T}) \leq -T^{1/6}) \\ &\leq (\mathrm{const})\ \Phi(-T^{1/6}) \leq \mathrm{e}^{-T^{1/3}/4}\,, \end{aligned}$$

where Φ is the standard normal distribution function. Finally, since the slow growth condition on M holds we have using Lemma 1 that

$$M\ P(\xi_T > (1+\epsilon_T)\mu_T) \leq M\ \mathrm{e}^{-T^{1/3}/4} = o(1)\,. \tag{5.4}$$

Next we treat the case $\xi_T < (1-\epsilon_T)\mu_T$. Choose $\beta_n = \epsilon_n n$ with ϵ_T obeying (5.3). Then an application of Corollary 1 in Appendix A shows that

$$\begin{aligned} P(\xi_T < (1-\epsilon_T)\mu_T) &= P(T_0 + Z_1 + \cdots + Z_{[(1-\epsilon_T)\mu_T]} > T) \\ &\sim P(Z_1+\cdots+Z_{[(1-\epsilon_T)\mu_T]} - [(1-\epsilon_T)\mu_T]\mu > T - [(1-\epsilon_T)\mu_T]\mu) \\ &\sim \mu_T\ P(Z > \epsilon_T T) = (\mathrm{const}) T P(Z > \epsilon_T T) \\ &\sim ((\mathrm{const})\left(\frac{MP(Z > b(MT))}{P(Z > \epsilon_T T)}\right)^{-1} = o(M^{-1}), \end{aligned}$$

due to the function $1/P[Z > x]$ being regularly varying with index α, and (5.3). (See [34, Proposition 0.8 (iii), page 23].) $\square$

We need another auxiliary result.

Lemma 5. *For all $\delta > 0$,*

$$M\ [b(MT)]^{-1}\ EX_{\xi_T}\, 1_{[X_{\xi_T} > \delta b(MT)]} \to 0 \quad \text{as } T \to \infty.$$

Proof. Choose $\epsilon_T \to 0$ such that (5.3) holds. Using Karamata's theorem, we have for large T,

$$\begin{aligned} & M[b(MT)]^{-1} \int_{\delta b(MT)}^{\infty} P(X_{\xi_T} > x,\, |\xi_T - \mu_T| \leq \epsilon_T \mu_T)\, dx \\ \leq\ & M\ [b(MT)]^{-1} \int_{\delta b(MT)}^{\infty} P\left(\max_{|i-\mu_T| \leq \epsilon_T \mu_T} X_i > x\right)\, dx \\ \leq\ & (\mathrm{const})\ \delta^{1-\alpha}\ \epsilon_T \to 0\,. \end{aligned}$$

Choose $c_T \to \infty$ such that $b(MT) = o(c_T^{-1}\epsilon_T T)$. It follows from the proof of Lemma 4 that

$$M\ P(|\xi_T - \mu_T| > \epsilon_T \mu_T) = o(c_T^{-\alpha})\,. \tag{5.5}$$

Let $K>0$ be a constant so large that $T^K > c_T b(MT)$ for large T. The following bound is straightforward:

$$\begin{aligned}
&\int_{\delta b(MT)}^{\infty} P(X_{\xi_T} > x,\, |\xi_T-\mu_T| > \epsilon_T\mu_T)\, dx \\
\leq & \quad \int_{\delta b(MT)}^{c_T b(MT)} P(|\xi_T-\mu_T|>\epsilon_T\mu_T)\, dx + \int_{c_T b(MT)}^{\infty} P(X_{\xi_T}>x,\, \xi_T<(1-\epsilon_T)\mu_T)\, dx \\
& + \int_{c_T b(MT)}^{T^K} P(\xi_T > (1+\epsilon_T)\mu_T)\, dx \; + \; \int_{T^K}^{\infty} P(X_{\xi_T}>x)\, dx \\
=: & \quad I_1+I_2+I_3+I_4\,.
\end{aligned}$$

Obviously, by (5.5),

$$M[b(MT)]^{-1} I_1 = (c_T-\delta)\; M\; P(|\xi_T-\mu_T|>\epsilon_T\mu_T) = o(1)\,.$$

Moreover, by Karamata's theorem,

$$\begin{aligned}
M[b(MT)]^{-1} I_2 \;&\leq\; M[b(MT)]^{-1}\int_{c_T b(MT)}^{\infty} P\left(\max_{i\leq(1-\epsilon_T)\mu_T} X_i > x\right)\, dx \\
&\leq\; (\text{const})\; M[b(MT)]^{-1}\,\mu_T \int_{c_T b(MT)}^{\infty} \overline{F}_{\text{on}}(x)\, dx \\
&\sim\; (\text{const})\; c_T^{1-\alpha}\, \frac{L_{\text{on}}(c_T b(MT))}{L_{\text{on}}(b(MT))} = (\text{const})\; c_T \frac{\overline{F}_{\text{on}}(c_T b(MT))}{\overline{F}_{\text{on}}(b(MT))}\,.
\end{aligned}$$

Using the right-hand inequality in Proposition 2 in Appendix B, with $x=c_T$, $t=b(MT)$ and $\varepsilon=\alpha-1-\delta>0$ for some small $\delta>0$, gives that there is a fixed t_0 such that for $x\geq 1$ and $t\geq t_0$

$$\frac{\overline{F}_{\text{on}}(c_T b(MT))}{\overline{F}_{\text{on}}(b(MT))} \leq (\alpha-\delta)\; c_T^{-(1+\delta)}\,.$$

This shows that $M[b(MT)]^{-1} I_2 \to 0$.

As for the proof of Lemma 4 (see (5.4)) we conclude that

$$M[b(MT)]^{-1} I_3 \leq M[b(MT)]^{-1}\int_{c_T b(MT)}^{T^K} \mathrm{e}^{\,-T^{1/3}/4}\, dx = o(1) \quad \text{as } T\to\infty$$

since the small growth condition on M holds. Using Markov's inequality and [16, Theorem 8.1 in Chapter I]), we have for $\epsilon\in(0,\alpha-1)$ and K sufficiently large,

$$M[b(MT)]^{-1} I_4 \leq EX_{\xi_T}^{\alpha-\epsilon} \int_{T^K}^{\infty} x^{-\alpha+\epsilon}\, dx = (\text{const})\; EX_{\xi_T}^{\alpha-\epsilon}\, T^{-K(\alpha-1-\epsilon)} = o(1)\,,$$

Combining all the estimates above, we finally proved the statement of the lemma. □

Now we are ready to deal with A_3.

Lemma 6. *As* $T\to\infty$,

$$[b(MT)]^{-1}[A_3 - EA_3] \xrightarrow{P} 0\,.$$

Proof. Fix $\delta > 0$. Define the iid random variables

$$\widehat{X}_T^{(m)} := \max\left(0, T^{(m)}_{\xi_T^{(m)}-1} + X^{(m)}_{\xi_T^{(m)}} - T\right) 1_{[\xi_T^{(m)} \geq 1]}$$

and their truncated versions

$$\widetilde{X}_T^{(m)} = \widehat{X}_T^{(m)} I_{[\widehat{X}_T^{(m)} \leq \delta b(MT)]}$$

By virtue of Lemma 5, it suffices to show that

$$[b(MT)]^{-1} \sum_{m=1}^{M} \left(\widetilde{X}_T^{(m)} - E\widetilde{X}_T\right) \xrightarrow{P} 0\,.$$

The variance of the sum on the left–hand side is given by

$$M[b(MT)]^{-2}\mathrm{Var}(\widetilde{X}_T) \leq M[b(MT)]^{-2} E\widetilde{X}_T^2\,,$$

and so it suffices to show that the right-hand side converges to zero. Assume $\epsilon_T \to 0$ satisfies (5.3). Then we have

$$\begin{aligned} E\widetilde{X}_T^2 &\leq \delta^2[b(MT)]^2\, P(|\xi_T - \mu_T| > \epsilon_T\mu_T) + \int_0^{\delta^2[b(MT)]^2} P(X_{\xi_T} > \sqrt{x},\, |\xi_T - \mu_T| \leq \epsilon_T\mu_T)\, dx \\ &=: I_1 + I_2\,. \end{aligned}$$

By Lemma 4 we have

$$M[b(MT)]^{-2} I_1 = o(1)\,.$$

An application of Karamata's theorem yields that

$$\begin{aligned} M[b(MT)]^{-2} I_2 &\leq M[b(MT)]^{-2} \int_0^{\delta^2[b(MT)]^2} P\left(\max_{|i-\mu_T|\leq\epsilon_T\mu_T} X_i > \sqrt{x}\right) dx \\ &\leq (\mathrm{const})\; M[b(MT)]^{-2}\epsilon_T\mu_T \int_0^{\delta^2[b(MT)]^2} \overline{F}_{\mathrm{on}}(\sqrt{x})\, dx \\ &\sim (\mathrm{const})\epsilon_T\delta^2 MT\overline{F}_{\mathrm{on}}(\delta b(MT)) \sim (\mathrm{const})\; \delta^{2-\alpha}\; \epsilon_T = o(1)\,. \end{aligned}$$

This completes the proof. □

5.4. **α-stable limits: one dimensional convergence.** In this section we show that the random variables $A_2 = A_2(T)$ weakly converge to a stable distribution as $T \to \infty$. This fact and the results of the previous section, together with a Slutsky argument, prove the convergence of the one dimensional distributions in Theorem 2.

Introduce the iid mean zero random variables

$$J_k^{(m)} := X_k^{(m)} - r_{\mathrm{off}} Z_k^{(m)} = r_{\mathrm{on}} X_k^{(m)} - r_{\mathrm{off}} Y_k^{(m)} = r_{\mathrm{on}}(X_k^{(m)} - \mu_{\mathrm{on}}) - r_{\mathrm{off}}(Y_k^{(m)} - \mu_{\mathrm{off}})\,,$$

where

$$r_{\mathrm{on}} := \mu_{\mathrm{off}}/\mu \quad \text{and} \quad r_{\mathrm{off}} := \mu_{\mathrm{on}}/\mu\,.$$

The tails of the J_k's are regularly varying: as $x \to \infty$

$$P(J_k > x) \sim \frac{r_{\mathrm{on}}^{\alpha} L_{\mathrm{on}}(x)}{x^{\alpha}} \quad \text{and} \quad P(J_k \leq -x) \sim \frac{r_{\mathrm{off}}^{\alpha_{\mathrm{off}}} L_{\mathrm{off}}(x)}{x^{\alpha_{\mathrm{off}}}}.$$

Write

$$S_{T,m} := \sum_{k=1}^{\xi_T^{(m)}} J_k^{(m)} .$$

The following decomposition will be useful:

$$A_2 \;=\; \sum_{m=1}^{M} S_{T,m} + r_{\text{off}} \sum_{m=1}^{M} T^{(m)}_{\xi_T^{(m)}} - r_{\text{off}} \sum_{m=1}^{M} T_0^{(m)} 1_{\{\xi_T^{(m)} \geq 1\}} =: A_{21} + A_{22} + A_{23} .$$

In what follows, we show that A_{21} has an α-stable limit whereas A_{22} and A_{23} are asymptotically negligible. By Lemma 3 it follows that $[b(MT)]^{-1} E A_{23} \to 0$. Following the argument for Theorem 5.3 in Chapter I of [16] or [35, page 47], we obtain

$$ET_{\xi_T} = T + E(T_0\, 1_{\{\xi_T \geq 1\}}). \tag{5.6}$$

By virtue of Lemma 2, we have for large T

$$\frac{M}{b(MT)}\, E(T_{\xi_T} - T) = \frac{M}{b(MT)} E(T_0\, 1_{\{\xi_T \geq 1\}}) \leq (\text{const})\, \frac{MT^2 \overline{F}_{\text{on}}(T)}{b(MT)} \to 0,$$

since $T^2 \overline{F}_{\text{on}}(T) \sim T^{2-\alpha} L(T)$, as $T \to \infty$. Hence,

$$[b(MT)]^{-1}(A_{22} - EA_{22}) = r_{\text{off}} [b(MT)]^{-1} \sum_{m=1}^{M} [(T^{(m)}_{\xi_T^{(m)}} - T) - E(T_{\xi_T} - T)] \xrightarrow{P} 0 .$$

Again using the argument for Theorem 5.3 in Chapter I of [16], we obtain

$$EA_{21} = M\ E\xi_T\ EJ_1 = 0 .$$

In the remainder of this section we prove that A_{21} has an α-stable limit. In [29, Theorem 8 in Chapter IV] one can find the following necessary and sufficient conditions for the sums of row-wise iid random variables $S_{T,m}$, $m = 1, \dots, M$, to converge weakly to an α-stable distribution $S_\alpha(c\,\sigma, \beta, 0)$: as $T \to \infty$

$$(A) \quad M\ P(S_{T,1} > x b(MT)) \to c^\alpha x^{-\alpha} \quad \text{for all } x > 0,$$

$$(B) \quad M\ P(S_{T,1} \leq -x b(MT)) \to 0 \quad \text{for all } x > 0,$$

$$(C) \quad \lim_{\epsilon \downarrow 0} \limsup_{T \to \infty} M\ [b(MT)]^{-2} \text{Var}(S_{T,1}\, 1_{\{|S_{T,1}| < \epsilon b(MT)\}}) = 0 .$$

See also [31, 34] for point process interpretations of these conditions.

We have $S_{T,1} = S^{(1)}(T) - S^{(2)}(T)$, where

$$S^{(1)}(T) = r_{\text{on}} \sum_{k=1}^{\xi_T} (X_k - \mu_{\text{on}}) \quad \text{and} \quad S^{(2)}(T) = r_{\text{off}} \sum_{k=1}^{\xi_T} (Y_k - \mu_{\text{off}}) .$$

Define

$$S_n = \sum_{k=1}^{n} J_k\,, \quad S_n^{(1)} = r_{\text{on}} \sum_{k=1}^{n} (X_k - \mu_{\text{on}})\,, \quad S_n^{(2)} = r_{\text{off}} \sum_{k=1}^{n} (Y_k - \mu_{\text{off}}) .$$

The proof of (A), (B) and (C) is now presented via a series of lemmas.

Lemma 7. *For all $x > 0$,*

$$M\ P(-S_{[\mu_T]}^{(1)} > xb(MT)) = o(1), \quad \text{as } T \to \infty.$$

Proof. Let $D := D_T$ be a positive function such that $D \to 0$ and as $T \to \infty$,

$$D\ M \to \infty \quad \text{and} \quad D\ b(MT) \to \infty\,. \tag{5.7}$$

Introduce the random variables

$$\widetilde{X}_k := X_k 1_{[X_k \le Db(MT)]} \quad \text{and} \quad \widetilde{S}_n^{(1)} := \sum_{k=1}^n (\widetilde{X}_k - E\widetilde{X})\,,$$

and assume without loss of generality that $r_{\text{on}} = 1$. We have

$$p(T) := P(-S_{[\mu_T]}^{(1)} > xb(MT)) \le P(-\widetilde{S}_{[\mu_T]}^{(1)} > xb(MT) - \mu_T E(X - \widetilde{X})).$$

Using Karamata's theorem, we have

$$\frac{\mu_T E(X - \widetilde{X})}{b(MT)} \sim (\text{const})\ \frac{D^{1-\alpha}}{M}\ \frac{L_{\text{on}}(Db(MT))}{L_{\text{on}}(b(MT))} = (\text{const})\ \frac{D}{M}\ \frac{\overline{F}_{\text{on}}(Db(MT))}{\overline{F}_{\text{on}}(b(MT))}\,. \tag{5.8}$$

Using the left-hand inequality of Proposition 2 in Appendix B, with $x = 1/D$, $t = Db(MT)$ and $\varepsilon = 2 - \alpha$ gives that there is a fixed t_0 such that for $x \ge 1$ and $t \ge t_0$ the right-hand side of (5.8) is bounded by

$$\frac{(\text{const})}{\alpha - 1}\ \frac{1}{DM}\,,$$

which is $o(1)$ by (5.7). So we may bound the probability $p(T)$ for large T from above by

$$p(T) \le P\left(-[\text{Var}(\widetilde{X})\mu_T]^{-1/2}\widetilde{S}_{[\mu_T]}^{(1)} > a_T(x)\right) \quad \text{where } a_T(x) = \frac{xb(MT)/2}{[\text{Var}(\widetilde{X})\mu_T]^{1/2}}.$$

Using a non-uniform Berry–Esséen estimate in the central limit theorem (see [30, Theorem 5.16]), the right-hand side is bounded by

$$\overline{\Phi}(a_T(x)) + (\text{const})\ \frac{E|\widetilde{X}|^3}{\mu_T^{1/2}[\text{Var}(\widetilde{X})]^{3/2}\ (1 + a_T(x))^3}\,, \tag{5.9}$$

where $\overline{\Phi}$ denotes the right tail of the standard normal distribution. Notice that

$$a_T(x) \sim (\text{const})\ \left(\frac{M}{D^{2-\alpha}}\ \frac{L_{\text{on}}(b(MT))}{L_{\text{on}}(Db(MT))}\right)^{1/2} = (\text{const})\ \left(\frac{M}{D^2}\ \frac{\overline{F}_{\text{on}}(b(MT))}{\overline{F}_{\text{on}}(Db(MT))}\right)^{1/2}\,.$$

As above we apply the left-hand inequality of Proposition 2 in Appendix B, to obtain that for large T

$$a_T(x) > (\text{const})\ (\alpha - 1)\ M^{1/2}\,,$$

so the first term in (5.9) decreases at an exponential (in M) rate and, hence, is $o(M^{-1})$. The second term behaves asymptotically as

$$(\text{const})\ T\ \frac{[Db(MT)]^{3-\alpha}L_{\text{on}}(Db(MT))}{[xb(MT)]^3} \sim (\text{const})\ x^{-3}\ \frac{D^3}{M}\ \frac{\overline{F}_{\text{on}}(Db(MT))}{\overline{F}_{\text{on}}(b(MT))}\,. \tag{5.10}$$

Using the left-hand inequality of Proposition 2 in Appendix B, as before, gives that the right-hand side of (5.10) is bounded from above by

$$\frac{(\text{const})}{\alpha-1}\, x^{-3}\, \frac{D}{M} = o(M^{-1}).$$

This completes the proof. □

Let $\epsilon_T \to 0$ and define the event

$$\Theta_T := \{|\xi_T - \mu_T| \le \epsilon_T \mu_T\}. \tag{5.11}$$

Lemma 8. *For all $x > 0$,*

$$M\ P(|S_{T,1} - S_{[\mu_T]}| > xb(MT),\, \Theta_T) = o(1) \quad \text{as } T \to \infty.$$

Proof. Using [30, Theorem 2.3], we have

$$\begin{aligned} P(|S_{T,1} - S_{[\mu_T]}| > xb(MT),\, \Theta_T) &\le P\Big(\max_{|j-\mu_T|\le \epsilon_T\mu_T} |S_j - S_{[\mu_T]}| > xb(MT)\Big) \\ &\le (\text{const})\ P(|S_{[\epsilon_T\mu_T]}| > xb(MT)/2). \end{aligned}$$

Applying the same result, we also see that

$$\begin{aligned} P(|S_{[\epsilon_T\mu_T]}| > xb(MT)/2) &= P(|S^{(1)}_{[\epsilon_T\mu_T]} - S^{(2)}_{[\epsilon_T\mu_T]}| > xb(MT)/2) \\ &\le (\text{const}) \sum_{i=1}^{2} P\left(S^{(i)}_{[\epsilon_T\mu_T]} - \widehat{S}^{(i)}_{[\epsilon_T\mu_T]} > xb(MT)/4\right), \end{aligned}$$

where $\widehat{S}^{(1)}$ and $\widehat{S}^{(2)}$ are independent copies of $S^{(1)}$ and $S^{(2)}$. Using Corollary 1 in Appendix A, we see that the two probabilities on the right-hand side multiplied by M are asymptotic to

$$\begin{aligned} &(\text{const})\ M\ \epsilon_T\mu_T\left[[xb(MT)]^{-\alpha} L_{\text{on}}(b(MT)) + [xb(MT)]^{-\alpha_{\text{off}}} L_{\text{off}}(b(MT))\right] \\ \sim\ &(\text{const})\ x^{-\alpha}\ \epsilon_T \to 0\,. \end{aligned}$$

This completes the proof. □

Lemma 9. *For all $x > 0$,*

$$M\ P(S_{[\mu_T]} \le -xb(MT)) = o(1) \quad \text{as } T \to \infty.$$

Proof. We have

$$P(S_{[\mu_T]} \le -xb(MT)) \le P(-S^{(1)}_{[\mu_T]} > xb(MT)/2) + P(S^{(2)}_{[\mu_T]} > xb(MT)/2).$$

The first probability is $o(M^{-1})$ by Lemma 7. The second probability can be treated as follows. Let $\delta > 0$ such that $\alpha + \delta < \alpha_{\text{off}}$. Using Markov's inequality and a bound for the $(\alpha+\delta)$th moment of sums of independent mean-zero random variables (see [29, p. 60]), we obtain

$$\begin{aligned} M\ P(S^{(2)}_{[\mu_T]} > xb(MT)/2) &\le (\text{const})\ M[xb(MT)]^{-\alpha-\delta} E|S^{(2)}_{[\mu_T]}|^{\alpha+\delta} \\ &\le (\text{const})\ \frac{M\mu_T}{[b(MT)]^{\alpha+\delta}}\ \frac{E|Y|^{\alpha+\delta}}{x^{\alpha+\delta}}, \end{aligned}$$

which is $o(1)$ since $T/(b(T))^{\alpha+\delta}$ is regularly varying with index $-\delta/\alpha$. □

In the following lemma we finally conclude that A_{21} converges to an α-stable limit.

Lemma 10. *Let c and σ be as in Theorem 2. Then*

$$[b(MT)]^{-1}A_{21} \stackrel{d}{\to} c\, X_{\alpha,\sigma,1}(1) \quad \text{as } T\to\infty.$$

Proof. We prove (A), (B) and (C); see the beginning of this section.

Proof of (A). We have to show that for all $x>0$

$$M\ P(S_{T,1} > xb(MT)) \to c^\alpha\, x^{-\alpha}, \quad \text{as } T\to\infty.$$

Recall the definition of the event Θ_T from (5.11). By virtue of Lemma 4 it suffices to consider the probability $P(S_{T,1} > xb(MT),\, \Theta_T)$. For $\delta\in(0,1)$ we have,

$$P(S_{T,1} > xb(MT),\, \Theta_T) \le P(S_{T,1} - S_{[\mu T]} > \delta xb(MT),\, \Theta_T) + P(S_{[\mu T]} > (1-\delta)xb(MT)).$$

The first probability is $o(M^{-1})$ by Lemma 8. Using Corollary 1 in Appendix A, we obtain

$$M\ P(S_{[\mu T]} > (1-\delta)xb(MT)) \sim r_{\rm on}^{\alpha}\mu^{-1}\,(1-\delta)^{-\alpha}x^{-\alpha}\,. \tag{5.12}$$

A lower bound is given by

$$\begin{aligned}
&P(S_{T,1} > xb(MT),\, \Theta_T)\\
&\ge\ P(S_{T,1} - S_{[\mu T]} > -\delta xb(MT),\, S_{[\mu T]} > (1+\delta)xb(MT)\,, \Theta_T)\\
&\ge\ P(S_{[\mu T]} > (1+\delta)xb(MT)) - P(S_{T,1} - S_{[\mu T]} \le -\delta xb(MT),\, \Theta_T) - P(\Theta_T^c).
\end{aligned}$$

The second and third probabilities are $o(M^{-1})$ by Lemmas 8 and 4. Then, using Corollary 1 in Appendix A, gives

$$M\ P(S_{[\mu T]} > (1+\delta)xb(MT)) \sim r_{\rm on}^{\alpha}\mu^{-1}\,(1+\delta)^{-\alpha}x^{-\alpha}\,. \tag{5.13}$$

The proof of (A) is completed by letting $\delta\to 0$ in (5.12) and (5.13).

Proof of (B). We prove that for all $x>0$

$$M\ P(S_{T,1} \le -xb(MT)) \to 0\,.$$

For $\delta\in(0,1)$ we have

$$P(S_{T,1} \le -xb(MT),\, \Theta_T) \le P(S_{T,1} - S_{[\mu T]} \le -\delta xb(MT),\, \Theta_T) + P(S_{[\mu T]} \le -(1-\delta)xb(MT)).$$

The first probability is $o(M^{-1})$ by Lemma 8 and so is the second one by virtue of Lemma 9. This completes the proof of (B).

Proof of (C). We show that

$$\lim_{\epsilon\downarrow 0}\limsup_{T\to\infty}\frac{M}{[b(MT)]^2}\mathrm{Var}(S_{T,1}1_{\{|S_{T,1}|<\epsilon b(MT)\}}) = 0.$$

We have

$$\mathrm{Var}(S_{T,1}1_{\{|S_{T,1}|<\epsilon b(MT)\}}) \le \int_0^{\epsilon^2[b(MT)]^2} P(S_{T,1} > \sqrt{x})\,dx + \int_0^{\epsilon^2[b(MT)]^2} P(S_{T,1} \le -\sqrt{x})\,dx.$$

In our proof we will only consider the first integral. The second one can be treated analogously. Obviously,

$$\begin{aligned}
\frac{M}{[b(MT)]^2}\int_0^{\epsilon^2[b(MT)]^2} P(S_{T,1} > \sqrt{x})\,dx \quad &\le \quad \epsilon^2 + \frac{M}{[b(MT)]^2}\int_{\epsilon^2[b(MT)]^2/M}^{\epsilon^2[b(MT)]^2} P(S_{T,1} > \sqrt{x})\,dx\\
&=: \quad \epsilon^2 + p(T,\epsilon)\,.
\end{aligned}$$

By virtue of Lemma 4 it suffices to replace in $p(T,\epsilon)$ the event $\{S_{T,1} > \sqrt{x}\}$ with $\{S_{T,1} > \sqrt{x}, \Theta_T\}$. Combining the upper bound in the proof of (A) with the argument in the proof of Lemma 8, we obtain the following bound for $p(T,\epsilon)$:

$$\frac{M}{[b(MT)]^2} \int_{\epsilon^2[b(MT)]^2/M}^{\epsilon^2[b(MT)]^2} \left[(\text{const})\ P(|S_{[\epsilon_T \mu_T]}| > \sqrt{x}/4) + P(S_{[\mu_T]} > \sqrt{x}/2)\right]\ dx\,.$$

As in the proof of Lemma 8, we can use a symmetrization inequality and Corollary 1 in Appendix A to show that the first term is $o(1)$ as $T \to \infty$. Another application of Corollary 1 in Appendix A yields

$$\begin{aligned}\frac{M}{[b(MT)]^2} \int_0^{\epsilon^2[b(MT)]^2} P(S_{[\mu_T]} > \sqrt{x}/2)\ dx \quad &\sim \quad (\text{const})\ \frac{M}{[b(MT)]^2}\ (\epsilon^2[b(MT)]^2)^{1-\alpha/2} L_{\text{on}}(b(MT)) \\ &\sim \quad (\text{const})\ \epsilon^{2-\alpha}.\end{aligned}$$

Now let ϵ go to zero to obtain the desired relation (C). □

This finishes the proof that the one dimensional distributions converge to a stable law.

5.5. **α-stable limits: finite dimensional convergence.** In this section we complete the proof of Theorem 2 by showing that the finite dimensional distributions of $A_{21} = A_{21}(T)$ converge to those of α-stable Lévy motion. We will only show convergence of the 2-dimensional distributions since the general case is analogous. The following lemma is the key to this convergence.

Lemma 11. *Let $b_1, b_2 \in \mathbb{R}$, $t_2 \ge t_1 \ge 0$. Define*

$$Z_T^{(1)} := b_1 \sum_{k=1}^{\mu_{Tt_1}} J_k^{(m)} \quad \text{and} \quad Z_T^{(2)} := b_2 \sum_{k=\mu_{Tt_1}+1}^{\mu_{Tt_2}} J_k^{(m)}.$$

Then we have as $T \to \infty$, for all $x > 0$

$$\begin{aligned} M\ P(Z_T^{(1)} + Z_T^{(2)} > xb(MT)) \quad &\sim \quad M\ P(Z_T^{(1)} > xb(MT)) + M\ P(Z_T^{(2)} > xb(MT)) \\ &\sim \quad \frac{r_{\text{on}}^\alpha}{\mu}\ [b_1^\alpha I_{[b_1>0]} t_1 + b_2^\alpha I_{[b_2>0]}(t_2 - t_1)]\ x^{-\alpha}\,. \end{aligned}$$

Proof. For $\delta \in (0, 0.5)$, we have

$$\begin{aligned} & P(Z_T^{(1)} > (1+\delta)xb(MT))\ P(|Z_T^{(2)}| \le \delta xb(MT)) \\ & + P(Z_T^{(2)} > (1+\delta)xb(MT))\ P(|Z_T^{(1)}| \le \delta xb(MT)) \\ \le\ & P(Z_T^{(1)} + Z_T^{(2)} > xb(MT)) \\ \le\ & \sum_{i=1}^{2} P(Z_T^{(i)} > (1-\delta)xb(MT)) +\ P(Z_T^{(1)} > \delta xb(MT))P(Z_T^{(2)} > \delta xb(MT))\,. \end{aligned}$$

Now the result follows from Corollary 1 in Appendix A by first letting $T \to \infty$ and then $\delta \to 0$. □

The next lemma establishes convergence of the 2-dimensional finite dimensional distributions of $(A(Tt), t \ge 0)$ by virtue of the results in Section 5.3 in combination with a Slutsky argument.

Lemma 12. *Let $b_1, b_2 \in \mathbb{R}$ and $t_2 \geq t_1 \geq 0$. Then, as $T \to \infty$,*

$$\begin{aligned} b_1 A_{21}(Tt_1) + b_2(A_{21}(Tt_2) - A_{21}(Tt_1)) \;&\stackrel{d}{\to}\; b_1\, c\, X_{\alpha,\sigma,1}(t_1) \\ &\quad + b_2\,(c\, X_{\alpha,\sigma,1}(t_2) - c\, X_{\alpha,\sigma,1}(t_1)). \end{aligned}$$

Proof. Define

$$S_{T,m} := b_1\,[b(MT)]^{-1} \sum_{k=1}^{\xi^{(m)}_{Tt_1}} J_k^{(m)} + b_2\,[b(MT)]^{-1} \sum_{k=\xi^{(m)}_{Tt_1}+1}^{\xi^{(m)}_{Tt_2}} J_k^{(m)}.$$

According to Petrov [29, Theorem 8 in Chapter IV], we have to show that as $T \to \infty$,

$$(A) \quad M\ P(S_{T,1} > x) \to c^\alpha [b_1^\alpha I_{[b_1>0]} t_1 + b_2^\alpha I_{[b_2>0]}(t_2 - t_1)] x^{-\alpha} \quad \text{for all } x > 0,$$

$$(C) \quad \lim_{\epsilon \downarrow 0} \limsup_{T\to\infty} M\ \mathrm{Var}(S_{T,1} 1_{\{|S_{T,1}|<\epsilon\}}) = 0\,.$$

We will only give the proof of (A). The proof of (C) follows in the same way as in the proof of Lemma 10. Let $\epsilon_T \to 0$ satisfy (5.3). Since we know from Lemma 4 that as $T \to \infty$,

$$M\ P(|\xi_{Tt_j} - \mu_{Tt_j}| > \epsilon_T \mu_{Tt_j}) = o(1), \quad j = 1, 2,$$

it suffices to consider in (A) the intersection of the event $\{S_{T,1} > x\}$ with

$$\Theta_T = \{|\xi_{Tt_j} - \mu_{Tt_j}| \leq \epsilon_T \mu_{Tt_j},\ j = 1, 2\}.$$

For $\delta \in (0, 1)$ we have

$$\begin{aligned} P(S_{T,1} > x,\ \Theta_T) \;&\leq\; P\left(b_1 \left[\sum_{k=1}^{\xi_{Tt_1}} J_k - \sum_{k=1}^{\mu_{Tt_1}} J_k\right] > \delta x\ b(MT)/2,\ \Theta_T\right) \\ &\quad + P\left(b_2 \left[\sum_{k=\xi_{Tt_1}+1}^{\xi_{Tt_2}} J_k - \sum_{k=\mu_{Tt_1}+1}^{\mu_{Tt_2}} J_k\right] > \delta x\ b(MT)/2,\ \Theta_T\right) \\ &\quad + P\left(b_1 \sum_{k=1}^{\mu_{Tt_1}} J_k + b_2 \sum_{k=\mu_{Tt_1}+1}^{\mu_{Tt_2}} J_k > (1-\delta) x\ b(MT)\right)\,. \end{aligned}$$

The first and second probabilities are $o(M^{-1})$ by Lemma 8. By Lemma 11, M times the third probability is asymptotic to

$$\begin{aligned} &M\ P\Big(b_1 \sum_{k=1}^{\mu_{Tt_1}} J_k > (1-\delta) x\ b(MT)\Big) + M\ P\Big(b_2 \sum_{k=1}^{\mu_{Tt_2}-\mu_{Tt_1}} J_k > (1-\delta) x\ b(MT)\Big) \\ \sim\;& \frac{r_{\mathrm{on}}^\alpha}{\mu}\ [b_1^\alpha I_{[b_1>0]} t_1 + b_2^\alpha I_{[b_2>0]}(t_2 - t_1)] x^{-\alpha}\,. \end{aligned}$$

A lower bound for $M\ P(S_{T,1} > x,\ \Theta_T)$ can be found in the same way as in the proof of Lemma 10. This completes the proof. $\square$

We finally established that the finite dimensional distributions of the processes $(A(Tt), t \geq 0)$ converge to those of the α-stable Lévy motion. This concludes the proof of Theorem 2.

6. FBM approximations for the infinite source Poisson model under fast growth

The next two sections relate fast connection rates associated with strong correlations of $N_T(\cdot)$ and fractional Brownian motion limits. Section 6 studies the infinite source Poisson model and the subsequent Section 7 considers the superposition of ON/OFF sources.

Recall that a mean-zero Gaussian process $(B_H(t), t \geq 0)$ with a.s. continuous sample paths is called *fractional Brownian motion* if it has covariance structure

$$\mathrm{Cov}(B_H(t), B_H(s)) = \frac{\sigma_H^2}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \quad \text{for some } \sigma_H > 0,\ H \in (0,1).$$

The case $H = 1/2$ corresponds to Brownian motion and, if $H \in (1/2, 1)$, the autocovariance function of the increment process $(B_H(t) - B_H(t-1))_{t=1,2,\ldots}$, so-called *fractional Gaussian noise*, satisfies relation (2.6), i.e. it exhibits LRD. If $\sigma_H = 1$ we call B_H *standard fractional Brownian motion.* For more properties of fractional Brownian motion we refer to the monograph [38].

6.1. The main result.

The following theorem is our main result under the fast growth condition.

Theorem 3. *If Condition 2 holds, then the process $(A(Tt), t \geq 0)$ describing the total accumulated input in $[0, Tt]$, $t \geq 0$, satisfies the limit relation*

$$\frac{A(T\cdot) - \lambda\mu_{\text{on}}T(\cdot)}{[\lambda T^3 \overline{F}_{\text{on}}(T)\sigma^2]^{1/2}} \stackrel{d}{\rightarrow} B_H(\cdot)\,,$$

Here $\stackrel{d}{\rightarrow}$ denotes weak convergence in $(\mathbb{D}[0,\infty), J_1)$, B_H is standard fractional Brownian motion, $H = (3-\alpha)/2$ and σ^2 is given by (6.6) below.

Remark. Notice that $H = (3-\alpha)/2 \in (0.5, 1)$. Hence the corresponding fractional Gaussian noise sequence of B_H exhibits LRD in the sense of (2.6). This is in contrast to Theorem 1 where the limiting process, α-stable Lévy motion, has independent increments.

In the rest of this section we provide the proof of Theorem 3. As for Theorem 1, the decomposition of Section 4.2 will be the key for deriving the Gaussian limit. As in Section 4 we give the proof in several steps. We use the same notation as in that section.

6.2. FBM limits: one dimensional convergence.

We show that when $\lambda(T)$ grows faster, so that Condition 2 holds and $b(\lambda T)/T \to \infty$, $(A(T) - \lambda\mu_{\text{on}}T)/\sigma_T(1)$ is asymptotically normal, where for $t > 0$ we define

$$\sigma_T^2(t) = \lambda(T)(Tt)^3\overline{F}_{\text{on}}(Tt). \tag{6.1}$$

For this, consider the decomposition and representation (4.2) and (4.5), which in particular gives that

$$\lambda\mu_{\text{on}}T = EA(T) = EA_1 + EA_2 + EA_3 + EA_4\,,$$

so that

$$\begin{aligned} A(T) - \lambda\mu_{\text{on}}T =& (A_1 - P_1 Ej_1) + (A_2 - P_2E(T-j_2) + (A_3 - P_3E(j_3+t_3)) + (A_4 - P_4T)) \\ &+ (P_1 - EP_1)Ej_1 + (P_2 - EP_2)E(T-j_2) + (P_3 - EP_3)E(j_3+t_3) \\ &+ (P_4 - EP_4)T. \end{aligned}$$

Here, since $P_i - EP_i = O_P(\sqrt{EP_i})$, $i = 1,2,3$, by the central limit theorem and the fact that $\mathrm{Var}(P_4) = E(P_4) \to 0$, it follows from (4.3), (4.7), (4.11) and (4.14), and straightforward calculation that

$$A(T) - \lambda\mu_{\mathrm{on}}T = (A_1 - P_1 Ej_1) + (A_2 - P_2 E(T - j_2) + (A_3 - P_3 E(j_3 + t_3)) + (A_4 - P_4 T)) + o_P(\sigma_T(1)), \tag{6.2}$$

The first three terms in (6.2) will be shown to be asymptotically normal and the fourth is of smaller order. We start by considering

$$A_1 - P_1 Ej_1 \stackrel{d}{=} \sum_{k=1}^{P_1} (j_{k,1} - Ej_1)$$

where P_1 is Poisson with mean λm_1. To see why A_1 is asymptotically normal, observe there are approximately λT iid summands. We check Lyapunov's condition ([14, page 286], [36, page 319]) for asymptotic normality of the sums

$$S_n = \sum_{k=1}^{n} (j_{k,1} - Ej_1)\,.$$

¿From (4.9), (4.10) we have with $\sigma_1^2 = \alpha/((2-\alpha)(3-\alpha))$:

$$\begin{aligned} \mathrm{Var}(S_{[\lambda T]}) &\sim \lambda T\, T^2 \overline{F}_{\mathrm{on}}(T)\, \sigma_1^2 = \sigma_T^2(1)\sigma_1^2, \\ L_{[\lambda T]}^{(3)} &:= \sum_{k=1}^{[\lambda T]} E|j_{k,1} - Ej_1|^3 \le (\mathrm{const})\ \lambda T\, T^3 \overline{F}_{\mathrm{on}}(T)\,. \end{aligned}$$

Therefore and by (3.3) we have for Lyapunov's ratio

$$\sigma_T^{-3}(1) L_{[\lambda T]}^{(3)} \to 0 \quad \text{as } T \to \infty.$$

Since Lyapunov's condition implies asymptotic normality, we get by the invariance principle for triangular arrays of iid random variables that

$$B_T(\cdot) := \sigma_T^{-1}(1)\ S_{[\lambda T\,\cdot]} \stackrel{d}{\to} B(\cdot)\sigma_1 \quad \text{in } \mathbb{D}[0,\infty),$$

where $B(\cdot)$ is a standard Brownian motion. We still have (4.17) at our disposal, so joint convergence holds:

$$\left(B_T(\cdot), \frac{P_1}{\lambda T}\right) \stackrel{d}{\to} (\sigma_1 B(\cdot), 1) \quad \text{in } D[0,\infty) \times \mathbb{R}.$$

We get after composing, that

$$(A_1 - P_1 Ej_1)/\sigma_T(1) \stackrel{d}{\to} N(0, \sigma_1^2) \tag{6.3}$$

It follows in a completely analogous way that

$$\begin{aligned} (A_2 - P_2 E(T - j_2))/\sigma_T(1) &\stackrel{d}{\to} N(0,\sigma_2^2) \\ (A_3 - P_3 E(t_3 - j_3))/\sigma_T(1) &\stackrel{d}{\to} N(0,\sigma_3^2). \\ (A_4 - P_4 T)/\sigma_T(1) &\stackrel{d}{\to} 0\,. \end{aligned} \tag{6.4}$$

Together, (6.2)–(6.4) show that

$$(A(T) - \lambda\mu_{\mathrm{on}}T)/\sigma_T(1) \stackrel{d}{\to} N(0,\sigma^2), \tag{6.5}$$

where

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \frac{\alpha}{(2-\alpha)(3-\alpha)} + \frac{2}{\mu_{\rm on}(3-\alpha)} = \frac{1}{3-\alpha}\Big[\frac{\alpha}{2-\alpha} + \frac{2}{\mu_{\rm on}}\Big]. \tag{6.6}$$

6.3. **FBM limits: finite dimensional convergence and tightness.** For convenience we write G_T for the quantity in Theorem 3, i.e.

$$G_T(t) = \frac{A(Tt) - \lambda\mu_{\rm on}Tt}{[\lambda T^3 \overline{F}_{\rm on}(T)\sigma^2]^{1/2}}.$$

Since

$$\sigma_T^2(t)/\sigma_T^2(1) = (tT)^3\overline{F}_{\rm on}(tT)/T^3\overline{F}_{\rm on}(T) \to t^{3-\alpha}$$

it follows as in (6.5) that the one dimensional distributions of G_T converge to those of B_H. Suppose now that the finite dimensional distributions of G_T were proved to be asymptotically jointly normal. Let $\{G(t), t \geq 0\}$ be a Gaussian process whose finite dimensional distributions match the weak limits of the finite dimensional distributions of $G_T(\cdot)$. We know that for each t, $G_T(t) \xrightarrow{d} G(t)$. We also know that for any $h > 0$

$$\begin{aligned}\{G_T(t+h) - G_T(h), t \geq 0\} =& \Big\{\int_0^{Tt} \big(N(s+Th) - \lambda\mu_{\rm on}\big)ds, t \geq 0\Big\} \\ \stackrel{d}{=}& \Big\{\int_0^{Tt} \big(N(s) - \lambda\mu_{\rm on}\big)ds, t \geq 0\Big\} \\ =& \{G_T(t), t \geq 0\},\end{aligned}$$

since $N(\cdot)$ is stationary. So for each T, $\{G_T(t), t \geq 0\}$ has stationary increments and hence so does $\{G(t), t \geq 0\}$. Therefore

$$\begin{aligned}\mathrm{Var}(G(t)) =& \mathrm{Var}\big(G(t+h) - G(h)\big) \\ =& EG^2(t+h) + EG^2(h) - 2EG(h)G(t+h)\end{aligned}$$

so that

$$\mathrm{Cov}\big(G(h), G(t+h)\big) = \frac{1}{2}\Big(EG^2(t+h) + EG^2(h) - \mathrm{Var}\big(G(t)\big)\Big)$$

and thus the covariance, and hence the finite dimensional distributions of $G(\cdot)$, are determined by the one dimensional marginal distributions of $G(\cdot)$. This means $G(\cdot) \stackrel{d}{=} B_H(\cdot)$.

This argument shows it is enough to prove that the finite dimensional distributions of $G_T(\cdot)$ are asymptotically normal. To show this, it is sufficient to show that the increments of $G_T(\cdot)$ are jointly asympotically normal. The proof uses the same methods as for one dimensional convergence and we hence only give a brief sketch.

Consider e.g. the joint distribution of $G_T(u)$ and $G_T(u+v) - G_T(u)$ for $u, v > 0$. By decomposing as in (4.1), both for T replaced by uT and by $(u+v)T$, and considering all intersections of the sets in the two decompositions, the problem is reduced to proving asymptotic joint normality of functions of the Poisson points in a number of disjoint sets. Since the sets are disjoint, and the functions hence independent, the sets may be considered separately. A typical such set is given by

$$R = \{(s,y) : 0 < s \leq uT, uT - s < y \leq (u+v)T - s\}$$

which contributes

$$A_u \stackrel{d}{=} \sum_{k=1}^{P} (T - X_k) 1_{[(\Gamma_k, X_k) \in R]}$$

to $G_T(u)$ and

$$A_v \stackrel{d}{=} \sum_{k=1}^{P} X_k 1_{[(\Gamma_k, X_k) \in R]}$$

to $G_T(u+v) - G_T(u)$, where P is a Poisson random variable with mean $\lambda m := \lambda \mathbb{L} \times F_{\rm on}(R)$. However, here the upper summation limits P may be replaced by λm using the same method as in Section 6.2, and asymptotic joint normality with non-random summation limits is straightforward. It follow that A_u and A_v are jointly asymptotically normal. Similar arguments for the other sets complete the proof of finite dimensional convergence of G_T to B_H.

To prove tightness of G_T in $D[0, K]$, for K fixed, we rewrite (4.2), with T replaced by $U = uT$ (and hence with $\{A_i, P_i\}$ defined from U instead of from T) and $0 \le u \le K$, as

$$A(U) - \lambda \mu_{\rm on} U = A(U) - EA(U) = \sum_{i=1}^{4} (A_i - EA_i), \tag{6.7}$$

with the aim to bound the fourth moments of the increments of G_T. Let c be a generic constant whose value may change from appearance to appearance. We first show that

$$E(A_1 - EA_1)^4 / \sigma_T^4(1) \le c u^2. \tag{6.8}$$

Now, with notation as in (4.5),

$$A_1 - EA_1 = A_1 - P_1 Ej_1 + P_1 Ej_1 - EP_1 Ej_1, \tag{6.9}$$

and

$$\begin{aligned} E(A_1 - P_1 Ej_1)^4 &= E\left(\sum_{k=1}^{P_1} (j_{k,1} - Ej_1)\right)^4 \\ &= E(E\{\left(\sum_{k=1}^{P_1} (j_{k,1} - Ej_1)\right)^4 |P_1\}) \\ &\le E(P_1^2 (E(j_1 - Ej_1)^2)^2 + P_1 E(j_1 - Ej_1)^4) \\ &\le c\{EP_1^2 (Ej_1^2)^2 + EP_1 Ej_1^4)\}. \end{aligned} \tag{6.10}$$

Recalling (4.3) and that $EP_1 = \lambda m_1$ (cf. also the derivation of (4.9)),

$$\begin{aligned} EP_1^2 (Ej_1^2)^2 / \sigma_T^4(1) &\le \{(\lambda m_1 + 1) Ej_1^2 / \sigma_T^2(1)\}^2 \\ &= \{(1 + \frac{1}{\lambda m_1}) \frac{U^3 \overline{F}_{\rm on}(U)}{T^3 \overline{F}_{\rm on}(T)} \int_0^1 \int_0^{1-s} y^2 \frac{F_{\rm on}(U dy)}{\overline{F}_{\rm on}(U)} ds\}^2. \end{aligned} \tag{6.11}$$

The left-hand inequality in Proposition 2 in Appendix B, with $x = 1/u, t = U$ gives that there is a fixed $u_0 > 0$ with $\overline{F}_{\rm on}(U)/\overline{F}_{\rm on}(T)$ bounded by a constant times $(U/T)^{-\alpha-\epsilon}$ for $\varepsilon = 2 - \alpha$ and $U > u_0$, so that then

$$\frac{U^3 \overline{F}_{\rm on}(U)}{T^3 \overline{F}_{\rm on}(T)} \le c \frac{U}{V} = u.$$

On the other hand, for $0 < U < u_0$,

$$\frac{U^3\overline{F}_{\rm on}(U)}{T^3\overline{F}_{\rm on}(T)} \le \frac{u_0^2}{T^2\overline{F}_{\rm on}(T)}\frac{U}{T} \le cu.$$

Since the double integral in (6.11) is bounded by a constant by Karamata's theorem, we obtain that for the case when $\lambda m_1 \ge 1$,

$$EP_1^2(Ej_1^2)^2/\sigma_T^4 \le cu^2. \tag{6.12}$$

Similarly, using also (3.3),

$$EP_1Ej_1^4/\sigma_T^4 \le c\frac{U^5\overline{F}_{\rm on}(U)}{\lambda T^6\overline{F}_{\rm on}(T)^2} \le c\frac{1}{\lambda T\overline{F}_{\rm on}(T)}\frac{U^5\overline{F}_{\rm on}(U)}{T^5\overline{F}_{\rm on}(T)} \le cu^2, \tag{6.13}$$

and, still assuming $\lambda m_1 \ge 1$, by (4.7),

$$\begin{aligned} E((P_1 - EP_1)Ej_1)^4/\sigma_T^4 &= (3(Ep_1)^2 + EP_1)(Ej_1)^4/\sigma_T^4 \\ &\le c(\lambda m_1)^2/\sigma_T^4 \le c\frac{U^2}{T^6\overline{F}_{\rm on}(T)^2} \le cu^2. \end{aligned} \tag{6.14}$$

Together, (6.9) - (6.14) show that (6.8) holds for $\lambda m_1 \ge 1$.

If instead $EP_1 = \lambda m_1 \le 1$ then also $EP_1^4 \le c$ and using that $0 \le j_1 \le U$ we obtain

$$E(A_1 - EP_1E(j_1))^4/\sigma_T^4 \le cU^4/\sigma_T^4 \le \frac{1}{(\lambda T\overline{F}_{\rm on}(T))^2}(\frac{U}{T})^4 \le cu^2. \tag{6.15}$$

Thus, (6.8) holds also in this case, and thus generally.

Calculations along the same lines give the same bounds as in (6.8) for $E(A_i - EA_i)^4/\sigma_T^4$ for $i = 2, \dots 4$. Since A_T has stationary increments, it then follows from (6.7) that

$$E(G_T(t+u) - G_T(t))^4 = EG_T(u)^4 \le cu^2,$$

for $0 \le u \le K, 0 \le t + u \le K$. By [4], Theorem 12.3, $\{G_T\}$ then is tight in the J_1 topology on $D[0, K]$. Since $K > 0$ is arbitrary and since we already have shown finite dimensional convergence, this proves Theorem 3.

7. FBM approximations for the superposition of ON/OFF processes under fast growth

In this section we assume that the fast growth Condition 2 holds. Define

$$d_T := [T^{3-\alpha}L_{\rm on}(T)M]^{1/2}.$$

By Lemma 1, Condition 2 is equivalent to $o(d_T) = T$, since

$$\frac{d_T}{T} = [M\,T\,\overline{F}_{\rm on}(T)]^{1/2}.$$

The sequence (d_T) will serve as the normalization in the central limit theorem for the total accumulated input $A(T)$ in $[0,T]$. This is intuitively clear from the fact that $A(T)$ is the sum of the M iid cumulative workload processes

$$G_T^{(m)} := \int_0^T (W_u^{(m)} - EW_u^{(m)})\,du\,, \quad m = 1, \dots, M\,,$$

each of which has variance (cf. [45])

$$\mathrm{Var}(G_T) \sim \sigma_0^2\,T^{3-\alpha}L_{\rm on}(T) \quad \text{as } T \to \infty\,, \tag{7.1}$$

where

$$\sigma_0^2 := \frac{2\,\mu_{\mathrm{off}}^2\,\Gamma(2-\alpha)/(\alpha-1)}{\mu^3\,\Gamma(4-\alpha)}\,.$$

7.1. **The main result.** Under the fast growth condition, the processes $(A(Tt), t \geq 0)$ have a fractional Brownian motion as limit.

Theorem 4. *If Condition 2 holds, then the processes $(A(Tt), t \geq 0)$ describing the cumulative input in $[0, Tt]$, $t \geq 0$, satisfy the limit relation*

$$\frac{A(T\cdot) - TM\mu^{-1}\mu_{\mathrm{on}}(\cdot)}{d_T} \stackrel{d}{\rightarrow} \sigma_0\, B_H(\cdot)\,. \tag{7.2}$$

Here $\stackrel{d}{\rightarrow}$ denotes weak convergence in $(\mathbb{D}[0,\infty), J_1)$ and B_H is standard fractional Brownian motion with $H = (3-\alpha)/2$.

7.2. **Proofs.** One dimensional convergence is established in the following lemma.

Lemma 13. *For every $t \geq 0$,*

$$d_T^{-1} \sum_{m=1}^{M} G_{Tt}^{(m)} \stackrel{d}{\rightarrow} N(0, \sigma_0^2 t^{3-\alpha}) \stackrel{d}{=} \sigma_0 B_H(t)\,. \tag{7.3}$$

Proof. In [30, Theorem 4.2] we find the following necessary and sufficient conditions for (7.3): as $T \to \infty$

$$(A) \quad M\; P(|G_{Tt}| \geq \epsilon\; d_T) \to 0 \quad \text{for all } \epsilon > 0,$$

$$(B) \quad M\; d_T^{-2}\,\mathrm{Var}(G_{Tt}\,1_{[|G_{Tt}| \leq \tau\; d_T]}) \to \sigma_0^2 t^{3-\alpha} \quad \text{for some } \tau > 0,$$

$$(C) \quad M\; d_T^{-1}\, E(G_{Tt}\,1_{[|G_{Tt}| \leq \tau\; d_T]}) \to 0 \quad \text{for some } \tau > 0.$$

(A) and (C) follow from the fact that $P(|G_T| \geq \epsilon\; d_T) = 0$ for large T, since $T = o(d_T)$ and $|G_T| \leq T$ a.s. The proof of (B) follows from the same observation in combination with (7.1). □

Now, it is only a small step to prove convergence of the finite dimensional distributions of A. We only consider 2-dimensional convergence, since the general case is completely analogous. We have to show that, for $b_1, b_2 \in \mathbb{R}$ and $t_2 \geq t_1 \geq 0$,

$$d_T^{-1} \sum_{m=1}^{M} [b_1 G_{Tt_1}^{(m)} + b_2 G_{Tt_2}^{(m)}] \stackrel{d}{\rightarrow} b_1\sigma_0 B_H(t_1) + b_2 \sigma_0 B_H(t_2)\,.$$

Again using [30, Theorem 4.2], one has to show the statements corresponding to (A)–(C) above. The proofs of (A) and (C) follow in the same way as in Lemma 13. For (B) we have to show that for $t_1 \leq t_2$, as $T \to \infty$

$$M d_T^{-2}\,\mathrm{Cov}(G_{Tt_1}, G_{Tt_2}) \to \frac{\sigma_0^2}{2}\,\left[t_1^{2H} + t_2^{2H} - (t_2 - t_1)^{2H}\right] = \mathrm{Cov}(\sigma_0 B_H(t_1), \sigma_0 B_H(t_2))\,. \tag{7.4}$$

But this follows from

$$\mathrm{Cov}(G_{Tt_1}, G_{Tt_2}) = \frac{1}{2}\left[\mathrm{Var}(G_{Tt_1}) + \mathrm{Var}(G_{Tt_2}) - \mathrm{Var}(G_{Tt_2} - G_{Tt_1})\right]\,,$$

the fact that G has stationary increments and (7.1). Therefore, the finite dimensional distributions of A converge to those of fractional Brownian motion.

It remains to show that the family of stochastic processes in (7.2) is tight in $(\mathbb{D}[0,K],J_1)$ for any fixed $K>0$. We will show that for small $u>0$ and $T\ge T^*$

$$E\Big|d_T^{-1}\sum_{m=1}^{M} G_{Tu}^{(m)}\Big|^2 \le (\mathrm{const})\; u^{1+\varepsilon}\,,$$

for some small $\varepsilon>0$. Then Theorem 12.3 in Billingsley [4] gives the result.

According to (7.1) we have for T large enough

$$E\Big|d_T^{-1}\sum_{m=1}^{M} G_{Tu}^{(m)}\Big|^2 = \frac{M}{d_T^2}\; EG_{Tu}^2 = \frac{EG_{Tu}^2}{T^{3-\alpha}L_{\mathrm{on}}(T)} \le 2\sigma_0^2\;\frac{EG_{Tu}^2}{EG_T^2}\,.$$

By (7.1) we know that the function EG_x^2 is regularly varying with index $3-\alpha$. Using the left-hand inequality of Proposition 2 in Appendix B, with $x=1/u$, $t=Tu$ and some small $\varepsilon>0$ such that $3-\alpha-2\varepsilon>1$, gives that there is a fixed t_0 such that for $u\le 1$ and $Tu\ge t_0$

$$\frac{EG_{Tu}^2}{EG_T^2} < \frac{1}{1-\varepsilon}\; u^{3-\alpha-\varepsilon}\,.$$

For $Tu<t_0$ we have for large enough T

$$\begin{aligned}
\frac{EG_{Tu}^2}{T^{3-\alpha}L_{\mathrm{on}}(T)} &\le \frac{(Tu)^2}{T^{3-\alpha}L_{\mathrm{on}}(T)}\\
&\le \frac{(Tu)^{1+\varepsilon}\; t_0^{1-\varepsilon}}{T^{3-\alpha}L_{\mathrm{on}}(T)}\\
&= \frac{T^{1-(3-\alpha-\varepsilon)}}{L_{\mathrm{on}}(T)}\; t_0^{1-\varepsilon}\; u^{1+\varepsilon}\\
&\le t_0^{1-\varepsilon}\; u^{1+\varepsilon}\,.
\end{aligned}$$

Since $3-\alpha-\varepsilon>1+\varepsilon$ we have for T large enough and $u\le 1$

$$E\Big|d_T^{-1}\sum_{m=1}^{M} G_{Tu}^{(m)}\Big|^2 \le \max(2\sigma_0^2/(1-\varepsilon),t_0^{1-\varepsilon})\; u^{1+\varepsilon}\,.$$

This completes the proof.

A. Large deviations of heavy-tailed sums

We present a large deviation result which is frequently used in the proof of Theorem 2. Let $(Z_k, k\ge 1)$ be iid random variables with distribution F such that

(A.1) $\qquad \overline{F}(x)=x^{-\alpha_1}L_1(x),\quad x>0\,,\quad$ for some $\alpha_1>0$ and L_1 slowly varying,

and denote by

$$S_n = Z_1+\cdots+Z_n\,,\quad n\ge 1\,,$$

the corresponding partial sums. Define

$$\mu_2(x)=x^{-2}\int_{|u|\le x} u^2\, dF(u)\,.$$

The following large deviation result was proved in [6].

Proposition 1. *Let $\beta_n \to \infty$ such that $S_n/\beta_n \xrightarrow{P} 0$. Suppose $B_n \subset [\beta_n, \infty)$. If the condition*

$$\lim_{n\to\infty} \sup_{x\in B_n} \left| n\ \mu_2(x)\ \ln(n\overline{F}(x))\right| = 0 \tag{A.2}$$

holds then

$$\lim_{n\to\infty} \sup_{x\in B_n} \left| \frac{P(S_n > x)}{n\overline{F}(x)} - 1\right| = 0. \tag{A.3}$$

Remark. Writing $M_n = \max_{k=1,\dots,n} Z_k$ for the partial maxima of the Z-sequence, we see that we can replace $n\overline{F}(x)$ in (A.3) by $P(M_n > x)$. This means that the large deviation $\{S_n > x\}$ is essentially due to the event $\{M_n > x\}$.

A consequence is the following result.

Corollary 1. *In addition to* (A.1) *assume that $EZ = 0$ and*

$$F(-x) = x^{-\alpha_2} L_2(x), \quad x > 0, \quad \text{for some } \alpha_2 > \alpha_1,\ \alpha_1 \in (1,2) \text{ and } L_2 \text{ slowly varying.}$$

Then (A.3) *holds with $\beta_n = a_n h_n$ and $B_n = [\beta_n, \infty)$ where (h_n) is any sequence with $h_n \uparrow \infty$ and (a_n) satisfies $n\overline{F}(a_n) \sim 1$.*

Proof. Since $(a_n^{-1} S_n)$ weakly converges to an α_1-stable distribution relation $\beta_n^{-1} S_n \xrightarrow{P} 0$ is immediate. Moreover, by Karamata's theorem

$$\mu_2(x) \le (\text{const})\ P(|Z| > x), \quad x > 0\,,$$

and so (A.2) is satisfied since

$$n\ \mu_2(x)\ \ln(n\ P(Z > \beta_n)) \le (\text{const})\ n\ P(|Z| > \beta_n)\ \ln(n\ P(|Z| > \beta_n)) \to 0\,.$$

This concludes the proof. $\square$

B. Bounds for regularly varying functions

Let $U(x)$ be a regularly varying function with index $\rho \in \mathbb{R}$, i.e. for $x > 0$

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{\rho}\,.$$

The following result can be found in Resnick [34], Proposition 0.8 (ii).

Proposition 2. *Take $\varepsilon > 0$. Then there is a fixed t_0 such that for $x \ge 1$ and $t \ge t_0$*

$$(1-\varepsilon)\ x^{\rho-\varepsilon} < \frac{U(tx)}{U(t)} < (1+\varepsilon)\ x^{\rho+\varepsilon}\,.$$

These bounds are called the *Potter bounds* in [5].

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