Fractionally Integrated Time Series

Classical ARIMA model:

$$(I - B)^d \Phi(B) X = \Theta(B)\epsilon,$$

where

X : Observed time series

ϵ : White noise (independent $N(0, \sigma^2)$)

B : Backshift Operator ($BX_t = X_{t-1}$ etc.)

d : differencing parameter (integral in classical approach)

$$\Phi(B) = \sum_{i=0}^{p} \phi_i B^i, \qquad \Theta(B) = \sum_{i=0}^{q} \theta_i B^i.$$

Fractional ARIMA model (Granger and Joyeux 1980, Hosking 1981):

d is fractional

$$(I - B)^d = I - dB + \frac{d(d-1)}{2} B^2 - ...$$

Spectral density:

$$
\begin{aligned}
f_X(\lambda) &= |I - e^{i\omega}|^{-d} \left| \frac{\Theta(e^{i\omega})}{\Phi(e^{i\omega})} \right|^2 \frac{\sigma^2}{2\pi} \\
&\sim c\lambda^{-2d} \text{ as } \lambda \downarrow 0.
\end{aligned}
$$

$-\frac{1}{2} < d < \frac{1}{2}$: Stationary and invertible

$0 < d < \frac{1}{2}$: "long memory"

$d = \frac{1}{2}$: "$\frac{1}{f}$ noise"

Fractional BM etc. satisfy same relations

Connection to Hurst H parameter: $d = H - \frac{1}{2}$.

Statistics

- ACF
- PCF
- Periodogram: Based on data $Y_1, ..., Y_N$,

$$I_N(\lambda) = \frac{1}{2\pi N} \left| \sum_{i=1}^{N} e^{i\lambda}(Y_i - \bar{Y}) \right|^2 .$$

Evaluate at *Fourier frequencies* $\lambda_{j,N} = \frac{2\pi j}{N}, \quad j = 1, 2, ..., \frac{N}{2}$.

Statistical properties (approximate): independent, exponentially distributed, $\mathrm{E}\{I_N(\lambda)\} \approx f(\lambda)$.

Plot $\log I_N(\lambda)$ *vs.* $\log \lambda$ for indication of long-memory behavior for small λ

Standard adjustments to PGM:

- *Tapering*
- *Smoothing*

Estimation of long-memory models

• Parametric: FARIMA models

• Non/semi-parametric: Fit a power-law model to lower tail of periodogram

  • *Least squares procedure* (Geweke and Porter-Hudak 1983)

  Linear regression of $\log I_N(\lambda)$ on $\log \lambda$ restricted to first m Fourier frequencies, where $m << \frac{N}{2}$ (bandwidth idea)

  • *Whittle estimation* (Künsch 1987)

  Given a parametric model $f(\lambda;\theta)$, choose θ to minimize

$$\sum_i \left\{ \log f(\lambda_{i,N};\theta) + \frac{I_N(\lambda_{i,N})}{f(\lambda_{i,N};\theta)} \right\}$$

  • Robinson (1995): Use Whittle (a.k.a. Gaussian) estimation procedure, but using power-law approximation to $f(\lambda;\theta)$ restricted to first m Fourier frequencies.

Robinson's procedure is justified by "standard" theory about approximations to perodiogram distribution, but applied in a non-standard way.

In principle, it is possible to improve Robinson's procedure to correct for bias resulting from standard assumptions in lower-tail of periodogram (Hurvich and Beltrao 1993). Robinson showed that this is not necessary for the asymptotics so long as $0 < d < \frac{1}{2}$, but this is important if $d \geq \frac{1}{2}$ (Hurvich and Ray 1995) or if the model is complicated by adding regression terms to the original model (Smith and Chen 1996)

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